Transform Approximations for Event Timing Models

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Abstract

Financial institutions are often exposed to credit-sensitive assets such as loans and corporate bonds. Positions involving these securities could be entered into for hedging, speculation or diversification purposes. An important consideration in the design and risk management of these credit-linked portfolios is the ability to calculate the distribution of the total loss. In this paper, we develop analytical approximations for the distribution of the total loss due to defaults in a loan portfolio at a fixed time horizon. We first consider a small-time expansion of the Laplace transform of the total loss; which we invert using the saddlepoint method. We handle specific cases for the loss specification and analyze approximation methods which are more suitable for longer time horizons. Our approximation method is generic in that it can handle a large class of models of default timing, and it addresses other important features of the loan portfolio, including stochastic volatility and state-dependent jumps at and between defaults. We illustrate our approximations for common types of portfolio loss specification. We also provide error bounds to our approximation and evaluate the methods by numerical simulation.

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1 Introduction

Financial institutions are exposed to several sources of corporate default risk. An institution needs to manage and reduce its exposure to credit-sensitive assets such as loans and corporate bonds. Corporate loan portfolios can be worth billions of dollars and an important consideration in the design and risk-management of these credit-linked portfolios is the ability to calculate the distribution for the total loss of the loan portfolio at a fixed time horizon.

In this paper, we develop analytical approximations for the total loss distribution of a loan portfolio consisting of corporate loans and bonds. We employ a small-time approximation method, which involves two steps. We first consider a suitable space-time Markov process associated with the portfolio loss at a future time horizon. The Markov process defines the joint-dynamics of the individual position losses, default counts and other state variables. The Laplace transforms of such Markov processes are seldom available in closed form. We thus use the infinitesimal generator of the Markov process to get a small-time Taylor expansion of the Laplace transform. We then invert the transform using the saddlepoint method (cf. Daniels [1954]) to get the distribution of portfolio loss. From this distribution, tail risk measures such as VaR can be easily computed. We also provide bounds for the compound and the individual approximation errors incurred in each of the above two steps, for a generic class of jump-diffusion default intensities.

There have been previous works which focus on computing the portfolio loss distributions of loan portfolios. Duffie and Pan [2001] consider a risk setting of multi-factor jump-diffusion dynamics for the default intensities and asset returns. They apply the delta-gamma approximation and compute the characteristic function for the change in the portfolio value, which they invert using a numerical Fourier inversion scheme as in Davies [1973]. Bruti-Liberati and Platen [2005] analyze strong approximations for pure jump diffusion processes. By using a jump-adapted time discretization scheme, they derive bias free expansions when the underlying stochastic differential equation (SDE) for the jump process is known. They also propose higher order discrete time strong approximations which are independent of the jump intensity. Unlike these previous papers, we provide a small-time expansion to the Laplace transform of the portfolio loss, and then obtain the distribution function using the saddlepoint method. This approximation is well suited for shorter time horizons and well diversified portfolios. It provides us with an analytical method which is computationally fast compared to traditional Monte Carlo simulations. This allows us to consider large portfolios consisting of thousands of names and extends itself to efficient scenario analysis.

In order to handle longer time horizons and badly diversified portfolios, we study other techniques of approximating the Laplace transform of the portfolio loss. We first focus on higher order expansions of the Laplace transform and then analyze the performance of combining several short term expansions. The saddlepoint method itself is well suited for computation of tail distributions, and hence we focus our efforts on effectively approximating the Laplace transform. All our methods considered in this work are generic, in that they can handle a large class of models of default timing and addresses other important features of the portfolio, including stochastic volatility and state-dependent jumps at and between defaults. These features are harder to capture with the traditional delta-gamma method and other traditional methods to analytically handle portfolio models.

The rest of the paper is organized as follows. Section 2 characterizes the loss of a loan portfolio and specifies the form of the loss distribution. Section 3 provides a motivating example for using a small-time expansion of the Cumulant Generating Function (CGF) of the portfolio loss. Section 4 derives the small-time expansion of the CGF associated with different Markov processes corresponding to the portfolio loss. Section 5 explains the computation of the loss distribution by the saddlepoint method. Section 6 analyzes the error incurred in the approximation. In Section 7, we describe techniques to improve the medium and long-term approximations to the CGF. In Section 8, we showcase our method by applying our generic intensity model to the doubly-stochastic and self-exciting models of default intensity. Section 9 evaluates our small-time analytical method by comparing it with an efficient numerical Monte Carlo algorithm. Section 10 concludes the paper. The Appendices contain proofs and technical information.

2 Preliminaries

2.1 Credit Portfolio Loss

In this section, we characterize the total loss of a loan portfolio at a time horizon H, comprising of debt in the form of corporate bonds and loans, referenced on a collection of names $j = 1, \ldots, n$. The portfolio is initiated at time $t = 0$, and its composition is held constant until time $t = H$. We fix the time horizon H under consideration to be shorter than the earliest maturity of the components of the portfolio, and calculate the losses incurred for each position. Let $\tau_j > 0, i \in 1, \ldots, n$ be random stopping times which are almost surely distinct – which correspond to the default times of assets $(1, \ldots, n)$ – relative to a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a right–continuous and complete information filtration $(\mathfrak{F}_t)_{t \geq 0}$. Since we are in a risk management framework, we consider $\mathbb{P}$ as the physical or the objective probability measure.

With each random stopping time τ_j, we associate an indicator process N^j, where $N_t^j = I(\tau_j \leq t)$, where $I(A)$ is the indicator function of an event $A \in \mathfrak{F}$. We also have the existence of a strictly positive, integrable and progressively measurable process λ_j such that the random variables

$$N_t^j - \int_0^t \lambda_j(s)(1 - N_s^j)ds \quad j \in \{1, \ldots, n\} \tag{1}$$

are martingales (refer Protter [2004]). The λ_j are correlated stochastic processes which determine the default rate of firm j. The process $\lambda_j(1 - N^j)$ is the $\mathfrak{F}_t$-conditional default rate of firm j, which we refer to as the intensity of firm j. The correlation structure of the λ_j determine the default dependence structure among the names in the portfolio.

The portfolio loss of a credit–sensitive portfolio at a fixed time horizon is given by

$$L_H = \sum_{j=1}^{n} l_j N_H^j \tag{2}$$

where l_j is a random variable which corresponds to the dollar value of the loss given default for firm j. We assume that each l_j is drawn independently of N^j, from a fixed distribution. In this paper, we provide an analytical approximation to the distribution of portfolio loss (L_H) at a fixed time horizon (H).

2.2 Portfolio Loss Distribution

Institutional loan portfolios have complex cash flow characteristics, which makes the distribution of the total loss very complex. We analytically approximate the distribution of L_H for some fixed time $H > 0$. To this end, we first expand the CGF of L_H and then invert the expanded CGF using the saddle point method. A similar approach is considered in Ait-Sahalia and Yu [2006], where the authors provide saddlepoint approximations to estimate the transition densities and cumulative distribution functions for Markov processes. Their method is applicable to models for which the Laplace transform can be obtained in closed form. The authors also show that useful approximations can be found by replacing the Laplace transform with its small time expansion. In this paper, we propose techniques to obtain analytical estimates of the Cumulant Generating Function (CGF), which we invert by applying the saddlepoint method to obtain the portfolio loss distribution.

CGF of the Portfolio Loss The Laplace transform of the portfolio loss is defined as

$$\psi(H,u) = \mathbb{E}\left(e^{uL_H}\right) \tag{3}$$

Given the characteristic function (defined as $\psi(H,\mathbf{i}u)$), the portfolio loss density function can be computed by Fourier Inversion as

$$p(H,y) = \frac{1}{2\pi\mathbf{i}}\int_{-\infty}^{\infty} e^{-\mathbf{i}uy}\psi(H,\mathbf{i}u)du \tag{4}$$

Alternatively, if $\psi(H,u) \in L^1(\mathbb{R})$, we can use the CGF ($K(H,u) = \log\psi(H,u)$) to rewrite Equation (4) as

$$p(H,y) = \frac{1}{2\pi\mathbf{i}}\int_{\hat{u}-\mathbf{i}\infty}^{\hat{u}+\mathbf{i}\infty} e^{K(H,u)-uy}du \tag{5}$$

where $\hat{u}$ is chosen to lie strictly within the region of convergence of $\psi(H,u)$. We obtain the portfolio loss density by expanding the CGF in the contour integral (Equation (5)) using a small-time approximation, and evaluating the integral using a saddlepoint approximation.

3 Motivating Example

In this section, we consider the simple case of Poisson default intensities and study the behavior of the Laplace transform $\psi(\Delta,u)$ and the CGF $K(\Delta,u)$ of the portfolio loss. The analysis in this section, motivates the approach of using a small-time expansion of the CGF. Let the default intensity of firm j be given by λ (a constant), which makes the firm defaults independent. We set the loss given default of all firms and the notionals to 1, i.e. $l = (1,\ldots,1)$. The total loss follows a binomial distribution given by

$$L_t = \sum_{j=1}^{n} l_j N_t^j$$

$$\sim B(n, 1 - e^{-\lambda t}) \tag{6}$$

and the CGF of the portfolio loss is given by

$$\begin{aligned} K(\Delta, u) &= \log(\psi(\Delta, u)) \\ &= n \log \left[e^{-\lambda \Delta} + (1 - e^{-\lambda \Delta}) e^{u} \right] \end{aligned} \tag{7}$$

Since the individual firm defaults are independent, the portfolio loss is also a Poisson process (a Markov process) with intensity $n\lambda$. Given the default indicator vector n_j for firm j at $t = 0$, we can Taylor expand the Laplace transform in Δ around $\Delta = 0$ to get the third order approximation

$$\psi^{(3)}(\Delta, u) = c_0 + c_1 \Delta + c_2 \Delta^2 + c_3 \Delta^3 + O(\Delta^4) \tag{8}$$

where

$$\begin{aligned} c_0 &= 1 \\ c_1 &= \sum_j \lambda (1 - n_j)(e^u - 1) \\ c_2 &= \frac{1}{2} \sum_j \lambda^2 (1 - n_j) \sum_k (1 - n_k)(e^u - 1)^2 \\ c_3 &= \frac{1}{3} \left[\sum_j \lambda^2 (1 - n_j) \sum_k (1 - n_k)(e^u - 1)^2 \right] \left[\sum_l \lambda (1 - n_l)(e^u - 1) \right] \\ &+ \frac{1}{3} \left[\sum_j \lambda^3 (1 - n_j) \sum_k (1 - n_k) \sum_l (1 - n_l)(e^u - 1)^3 \right] \end{aligned} \tag{9}$$

For the case considered in this section – a) no defaults at time zero $n_j = 0, \forall j \in \{1, \ldots, n\}$, b) constant Poisson intensity λ for all firms – we can write Equation (8) as

$$\psi^{(3)}(\Delta, u) = 1 + n\lambda(e^u - 1)\Delta + \frac{1}{2} n\lambda^2 (e^u - 1)^2 \Delta^2 + \frac{1}{3} n\lambda^3 (e^u - 1)^3 \Delta^3 + O(\Delta^4) \tag{10}$$

The third order expansion to the CGF can thus be computed as

$$\begin{aligned} K^{(3)}(\Delta, u) &= \log(\psi^{(3)}(\Delta, u)) \\ &= c_1 \Delta + \left(\frac{c_1^2}{2} - c_2 \right) \Delta^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{3} \right) \Delta^3 \\ &= n\lambda(e^u - 1)\Delta - \frac{1}{2} n\lambda^2 (e^u - 1)^2 \Delta^2 + \frac{1}{3} n\lambda^3 (e^u - 1)^3 \Delta^3 \end{aligned} \tag{11}$$

From Equation (7), for small values of $\lambda\Delta$, we can rewrite the terms as

$$K(\Delta, u) = n \log \left[e^{-\lambda \Delta} + (1 - e^{-\lambda \Delta}) e^u \right]$$

$$
\begin{aligned}
&\approx n\log(1-\lambda\Delta+\lambda\Delta e^u)\\
&= n\log(1+\lambda\Delta(e^u-1)) \qquad (12)\\
&= n\lambda(e^u-1)\Delta - \frac{1}{2}n\lambda^2(e^u-1)^2\Delta^2 + \frac{1}{3}n\lambda^3(e^u-1)^3\Delta^3 + O(\Delta^4)
\end{aligned}
$$

From the above equation, it is clear how the 3-rd order small-time CGF approximation matches the actual CGF very well, when $\lambda\Delta$ is small. To compute the error order, we apply a Taylor series expansion to the CGF and we have

$$
\begin{aligned}
K(\Delta,u) &= n\lambda(e^u-1)\Delta - \frac{1}{2}n\lambda^2 e^u(e^u-1)\Delta^2\\
&+ \frac{1}{6}n\lambda^3 e^{2u}(e^u-1)\Delta^3 + O(\Delta^4)\\
&= K^{(3)}(\Delta,u)(1+O(\Delta^2)) \qquad (13)
\end{aligned}
$$

In Appendix G, we show a different method to derive the result in Equation (13). Figure [1] shows the behavior of the approximate expansion for the CGF $K(\Delta, \mathbf{i}u)$ for increasing values of the order. We compare the analytical expression of the CGF of a binomial distribution as in Equation (7) with the order-2 and order-3 approximate expansion of the CGF. We fix $n = 100$, $\lambda = 5/365$, and $\Delta = 0.5$ in the figure. From the figure we see that the order-3 approximation comes close to the actual analytical expansion of the CGF. We see the effect of adding higher order terms in the figure. Figure [2] shows the inverse CDF for a binomial distribution (computed analytically) with the parameters as specified above, and the approximations obtained using a Fourier inversion of the order-2 and order-3 CGFs. We see that the small-time approximation closely matches the analytical distribution, with the order-3 approximation performing better that the order-2 approximation.

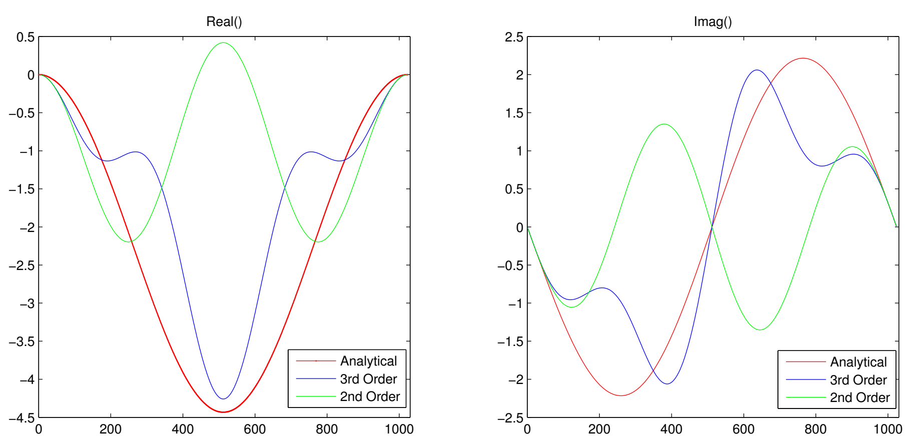

Figure 1: Behavior of the real and imaginary parts of $K(\Delta, \mathbf{i}u)$ for the analytical and the approximate expansions of the binomial CGF

We thus see that using a small-time expansion of the CGF, we obtain useful insights into the portfolio loss distribution.

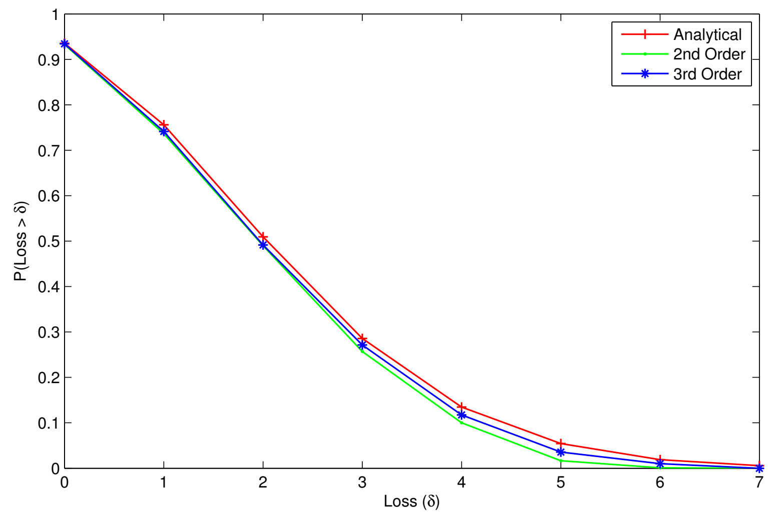

Figure 2: Comparison of the Inverse CDF for the binomial distribution with the order-3 and order-2 approximations of the CGF

4 Approximating the CGF

Motivated by the approach in Section 3, we now provide a general framework for approximating the CGF of the portfolio loss process.

4.1 Markov Process for the Portfolio Loss

In order to apply the saddlepoint approximation, we first need to consider a suitable space-time Markov process (t, Z) associated with the portfolio loss process L. In Section 4.3, we show that we get a Markov process for the portfolio loss by considering $Z = (L, Z^X)$, where Z^X consists of the state variables driving the default intensity. If the portfolio loss process L is already Markov (e.g. in case of a compound Poisson process), we have $Z = L$. An alternative is to consider the approach as in Giesecke et al. [2010a], where the authors prove that there exits a continuous-time Markov chain $M = (M^1, \ldots, M^n) \in \mathbb{S} = \{0, 1\}^n$ such that $M_t = N_t$ in distribution $\forall t \geq 0$, where $N_t = (N_t^1, \ldots, N_t^n)$ is the default indicator vector. The *mimicking* chain M has no joint transitions in any of its components, and each component M^j starts at 0. We denote the transition density of M^j by $\pi^j(\cdot, B), B \in \mathbb{S}$. If each of the default loss components (l_j) is drawn independently of N from a fixed distribution, the problem of computing the distribution of L_T is equivalent to computing the distribution of J_T where

$$J_T = \sum_{i=1}^{n} l_i M_T^i \tag{14}$$

The jump process J_T has an intensity given by

$$\lambda(t, M_t) = \sum_{i=1}^{n} \pi^i(t, M_t) \tag{15}$$

The choice of either of these methods of obtaining a Markov structure for the portfolio loss is driven by the computational complexity of constructing the corresponding Markov Chain as explained in Section 4.3.2. In the derivation of the small-time approximation in Section 4.2, we assume the presence of a Markov Process Z associated with the portfolio loss, and the existence of its infinitesimal generator, denoted by $\mathcal{A}$.

4.2 Small Time Approximation

The small-time approximation is obtained using a Taylor expansion of the CGF, which involves repeated application of the infinitesimal generator $\mathcal{A}$. If $h(t, Z)$ is a C^{K+2} function, we have its small-time Taylor Series expansion as (see Hille and Phillips [1957], Theorem 11.7.3 & 11.7.4)

$$\mathbb{E}[h(\Delta, Z_\Delta)|Z_0 = z] = \sum_{k=0}^{K} \frac{\Delta^k}{k!} \mathcal{A}^k h(0, z) + O(\Delta^{K+1}) \tag{16}$$

We apply the Taylor Series in Equation (16) in Δ to the function $h(t, z) = e^{uz_L}$ where z_L is the loss component (L) of z. Using this expansion we can compute an approximation to the Laplace transform of the portfolio loss as

$$\psi(\Delta, u|z) = \mathbb{E}(e^{uL_\Delta}|Z_0 = z) \approx \sum_{k=0}^{K} \frac{\Delta^k}{k!} \mathcal{A}^k e^{uz_L} \tag{17}$$

The following proposition proves a technical condition that is needed for the expansion in Equation (17) to be legitimate.

Proposition 4.1 *e^{uz_L} is in the domain $D_\mathcal{A}$ of the infinitesimal generator $\mathcal{A}$.*

Proof Proof in Appendix A

The n^{th} order approximation to the CGF $K^{(n)}$ is thus given by the expansion of $\log(\sum_{k=0}^{n} \frac{\Delta^k}{k!} \mathcal{A}^k e^{uz_L})$ which can be written as

$$K^{(n)}(\Delta, u|z) = uz_L + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta + \frac{1}{2} \frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta^2 + \cdots + O(\Delta^{n+1}) \tag{18}$$

The coefficients in Equation (18) can be evaluated once the Markov Process (Z), and hence the infinitesimal generator $(\mathcal{A})$ are exactly specified. In Section 4.3, we explicitly construct a Markov Process Z, by considering a generic jump diffusion intensity specification and derive the small time expansion for the CGF.

4.3 Jump Diffusion Intensities

In this section, we model the dynamics of the default intensities using a generic jump-diffusion framework and derive the small-time approximation to the CGF

4.3.1 Default Intensities

We assume that the default indicator process N^j of firm j admits intensity processes λ_j under $\mathbb{P}$ satisfying $\int_0^t \lambda_j(s)ds < \infty$ a.s. Thus the process defined by

$$N_t^j - \int_0^t \lambda_j(s)(1 - N_s^j)ds \tag{19}$$

is a martingale. The intensities have the following dynamics

$$d\lambda_j(t) = \mu_j(t)dt + \sum_i \sigma_{j,i}(t)dW_i(t) + \sum_m dJ_m(t) + \sum_o \xi_{j,o}(t)dN_t^o \tag{20}$$

where the W_is are independent Wiener processes and the J_ms are jump processes with intensities ι_m and jump size distributions ν_m. The intensity $\iota_m(t)$ takes the form $F_m(\Lambda(t))$ for some function F_m, where Λ is the vector of intensity processes λ_j. The term $\xi_{j,o}$ represents the sensitivity of firm $j's$ intensity to default of firm o. We assume enough regularity conditions on the function F_m, the drift μ_j, volatility $\sigma_{j,i}$ and the jump coefficients $\xi_{j,o}$ to guarantee a unique strong solution to the SDE in Equation (20).

The jump-diffusion dynamics of λ_j are fairly general, incorporating jumps in the intensity process of a single asset, diffusive movements between defaults, and the influence of past defaults on future intensities. A wide class of intensity models considered in literature fall in this category (see Errais et al. [2010], Duffie et al. [2000], Eckner [2009], Jarrow and Yu [2001], Papageorgiou and Sircar [2007], Feldhutter [2007], Chava and Jarrow [2004] and others). In this sense, our assumptions about the intensity do not significantly restrict the applicability of our results.

4.3.2 Small Time Approximation

We now consider a suitable Markov process associated with the portfolio loss in order to derive the explicit form of the infinitesimal generator.

Using an Augmented Markov Process Suppose the space-time process (t, Z_t^X) is a strong Markov process, where $Z^X = (\Lambda, N)$. A sufficient condition for this property is that (t, Z_t^X) is a regular affine process, see Filipovic [2004]. We note that the loss process L_t is not always a Markov process. However we have

Proposition 4.2 *If the space-time process (t, Z_t^X) is a strong Markov process, then the space-time process (t, Z_t) with the augmented vector $Z = (\Lambda, N, L)$, is also a strong Markov process for which we can find its infinitesimal generator $\mathcal{A}$*

Proof Proof in Appendix A

For the pure jump portfolio loss process L_t (refer Equation (2)) we have the following

$$dL_t = \sum_{0<s\le t} [L_s - L_{s-}] \tag{21}$$

where the precise specification of the jump terms will follow from a specification of the coefficients of the default intensity. For any bounded C^2 function h with bounded first and second derivatives, using Dynkin's formula, we get the time dependent generator as

$$\begin{aligned}\mathcal{A}h(t,z) = &\, h_t(t,z) + h_z(t,z)\mu(t,\lambda,n)\\ &+ \frac{1}{2}tr[h_{zz}(t,z)\sigma(t,\lambda,n)\sigma(t,\lambda,n)^T]\\ &+ \sum_m F_m(\lambda)\int_{\mathbb{R}^K}[h(t,z+z_m^{(1)}) - h(t,z)]d\nu_m(\epsilon)\\ &+ \sum_j \lambda_j(z)(1-n_j)[h(t,z+z_j^{(2)}) - h(t,z)]\end{aligned} \tag{22}$$

where

$$z_m^{(1)} = \begin{cases} 0 & \text{for all } l \text{ components} \\ \epsilon_j & \text{for all } \lambda \end{cases}$$

and

$$z_j^{(2)} = \begin{cases} 0 & \text{for all } \lambda_j, \text{ and } n_l \text{ where } l \neq j \\ \xi_{j,l}(t) & \text{for } \lambda_l \text{ where } l \neq j \\ 1 & \text{for } n_j \\ \sum_k g^{j,k}(t,\lambda) & \text{for } l \end{cases}$$

where the explicit form of the function $g^{j,k}$ can be computed from the actual intensity dynamics. The n-th order approximation to the CGF can be found explicitly by performing the Taylor expansion as in Equation (18). We thus have

Proposition 4.3 *The first and second order approximations to the CGF of L_t are given by*

$$\begin{aligned}K^{(1)}(\Delta,u|z) &= uz_L + c_1\Delta\\ K^{(2)}(\Delta,u|z) &= uz_L + c_1\Delta + \left(\frac{c_1^2}{2} - c_2\right)\Delta^2\end{aligned} \tag{23}$$

where $c_1 = \frac{\mathcal{A}e^{uz_L}}{e^{uz_L}}$ and $c_2 = \frac{\mathcal{A}^2 e^{uz_L}}{2e^{uz_L}}$ are given in Appendix C.

Proof Proof in Appendix B

Using a Mimicking Markov Chain A markov structure for the portfolio loss can also be obtained using a *mimicking* Markov Chain as in Section 4.1. This can be used to obtain the small-time expansion of the CGF of the portfolio loss. We also show in this section, that the infinitesimal generator of the *mimicking* Markov Chain is special case of the generator [refer Equation (22)] of the augmented process Z.

We have that L_H and J_H (refer Equation (15)) have the same distribution for each fixed time H. If the computation of the *mimicking* chain M is computationally feasible, we can, in lieu of Equation (17), opt to perform a small time expansion of the Laplace transform in terms of $\mathcal{A}_M$, which is the generator of the Markov Chain M. We then have

$$\begin{aligned}\psi(\Delta, u|J_0 = j) &= \mathbb{E}(e^{uJ_\Delta}|j) \\ &\approx \sum_{k=0}^{K} \frac{\Delta^k}{k!} \mathcal{A}_M^k e^{uj} \end{aligned} \tag{24}$$

As explained in Giesecke et al. [2010a], for a suitable real-valued function g on $\{0,1\}^n$, the generator $\mathcal{A}_M$ at time t is given by

$$\mathcal{A}_M g(B) = \sum_{i=1}^{n} \pi^i(t, B)(g(B + b_i) - g(B)) \qquad B, b_i \in \{0,1\}^n \tag{25}$$

where b_i has all components zero except the i^{th} component, which is one. The transition density $\pi^i(t, B)$ is given by

$$\begin{aligned}\pi^i(t, B) &= \mathbb{E}(\lambda_i(t)\mathbb{I}(\tau^i > t)|N_t = B) \\ &= \mathbb{E}(\lambda_i(t)(1 - n_i)|N_t = B), \qquad B \in \{0,1\}^n \end{aligned} \tag{26}$$

Comparing Equation (25) with Equation (22) we see that generator for the Markov chain $\mathcal{A}_M$ is just a specific case of the generator of the augmented Markov process $\mathcal{A}$. To illustrate the use of a mimicking Markov Chain, we consider the setting as in Section 3. In this case, $\pi^i(t, B)$ can be written as

$$\pi^i(t, B) = \lambda(1 - B^i) \tag{27}$$

where B^i is the i^{th} component of the vector B. From Equation (24) we have the following for the order-3 approximation $\psi^{(3)}(\Delta, u|j)$ of the Laplace transform

$$\begin{aligned}\psi^{(3)}(\Delta, u|j) &= \sum_{k=0}^{3} \frac{\Delta^k}{k!} \mathcal{A}_M^k e^{uj} \\ &= 1 + n\lambda(e^u - 1)\Delta + \frac{1}{2}n\lambda^2(e^u - 1)^2\Delta^2 + \frac{1}{3}n\lambda^3(e^u - 1)^3\Delta^3 \end{aligned} \tag{28}$$

Comparing this expression with Equation (10), we see that the mimicking chain method is an alternative way to compute the Laplace transform and hence the CGF. For a general jump-diffusion intensity setting, the choice between these methods to compute the CGF is governed by the computational complexity of evaluating the mimicking Markov Chain M. If M can be quickly computed, the simplified form of the generator $\mathcal{A}_M$ leads to simplified expressions for higher order expansions of the CGF.

5 Saddlepoint Approximation

To get the distribution of the portfolio loss, we must invert the Fourier transform as in Equation (5). The saddlepoint approximation method, first introduced by Daniels [1954], is a method to efficiently evaluate the contour integral. It is a method of steepest descent, that expands the contour integral around the unique minimizer of the CGF ($\hat{u}$) computed numerically (the CGF is convex under mild regularity conditions). The region outside the neighborhood of the saddlepoint does not contribute much to the integral and hence the density can be efficiently computed by Taylor expanding the exponent $K(\Delta, u|z) - u \cdot y$ around the unique minimizer $\hat{u}$ which satisfies

$$\left.\frac{\partial K(\Delta, u|z)}{\partial u}\right|_{u=\hat{u}} = y \tag{29}$$

Depending on the Markov process under consideration, different saddlepoint approximations can be obtained by choosing different leading terms in the local approximation, using a change of variables argument to the original Gaussian leading term expansions as in Daniels [1954] or Lugannani and Rice [1980], as shown in Wood et al. [1993]. In our case, the Markov process for the portfolio loss is a jump-diffusion process, and hence we choose to obtain the distribution using a non-Gaussian leading term for the saddlepoint expansion which corresponds to the following Laplace transform

$$e^{K_0(\Delta,u|z)} = e^{uz_L}\left[1 - \Delta\sum_j \lambda_j(1-n_j) + \Delta\sum_j \lambda_j(1-n_j)e^{u\sum_k g^{j,k}}\right] \tag{30}$$

which is an order-1 in Δ approximation to the real $K(\Delta, u|z)$. Another interpretation of this expression is that of the Laplace transform of a process that can have at most one jump in an interval of length Δ and the jump probability is $1 - \lambda\Delta$. Replacing the true cumulant generating function K by the n^{th} order approximation $K^{(n)}$ (refer Equation (18)), and using the non-Gaussian leading term defined above, we obtain the saddlepoint approximation of the cumulative distribution function (CDF). The construction of the saddlepoint approximation for the CDF follows a method proposed by Lugannani and Rice [1980]. We first start with the Fourier inversion formula, applied to the inverse CDF

$$\mathbb{P}(L_\Delta > a|Z_0 = z) = \frac{1}{2\pi\mathbf{i}}\int_{\hat{u}-\mathbf{i}\infty}^{\hat{u}+\mathbf{i}\infty} \exp\left(K(\Delta, u|z) - ua\right)\frac{du}{u} \tag{31}$$

where

$$a = \frac{\partial}{\partial u}K^{(n)}(\Delta, \hat{u}|z)$$

To incorporate the non-Gaussian leading term as defined above, we now define an implicit change of variable $u(w)$, so that we find an approximation to the function $K(\Delta, u|z) - ua$ that valid both when u is in a neighborhood of $\hat{u}$ and when u is in a neighborhood of 0. We have

$$K(\Delta, u|z) - ua = \frac{1}{2}w^2 - w\hat{w} \tag{32}$$

where

$$a = \frac{\partial}{\partial u}K_0(\Delta, \hat{w}|z)$$

Changing the variable from u to w as defined in Equation (32), we have

$$\mathbb{P}(L_\Delta > a|Z_0 = z) = \frac{1}{2\pi\mathbf{i}}\int_{\hat{w}-\mathbf{i}\infty}^{\hat{w}+\mathbf{i}\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right)\left(\frac{1}{u(w)}\frac{du(w)}{dw}\right)dw \tag{33}$$

Near $u = w = 0$, $K(\Delta, u|z) \approx u \times \frac{\partial K(\Delta,0|z)}{\partial u}$ and hence we have

$$w \approx \begin{cases} \left(\frac{a - \partial K(\Delta,0|z)/\partial u}{\hat{w}}\right) u & \text{if } a \neq \frac{\partial K(\Delta,0|z)}{\partial u} \\ \left(\frac{\partial^2 K(\Delta,0|z)}{\partial^2 u}\right) u & \text{otherwise} \end{cases} \tag{34}$$

Thus, near 0 we have

$$\frac{1}{u}\frac{du}{dw} \approx \frac{1}{w}$$

We now isolate the singularity in $u^{-1}du/dw$ near $w = 0$ by decomposing the integral in Equation (33) as

$$\begin{aligned} \mathbb{P}(L_\Delta > a | Z_0 = z) &= \frac{1}{2\pi \mathbf{i}} \int_{\hat{w}-\mathbf{i}\infty}^{\hat{w}+\mathbf{i}\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right) \frac{dw}{w} \\ &+ \frac{1}{2\pi \mathbf{i}} \int_{\hat{w}-\mathbf{i}\infty}^{\hat{w}+\mathbf{i}\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right) \left(\frac{1}{u}\frac{dw}{w} - \frac{1}{w}\right) dw \end{aligned} \tag{35}$$

The first integral in Equation (35) is equal to $1 - F_0(a|z)$, where $F_0(a|z)$ is the distribution function of the loss. Since $\frac{1}{u}\frac{dw}{w} - \frac{1}{w}$ is analytic in a neighborhood of zero, performing a Taylor expansion around $\hat{w}$ we get the following leading term

$$\frac{1}{u}\frac{dw}{w} - \frac{1}{w} = \frac{1}{\hat{u}}(K_0''(\hat{w}))^{1/2} \left(\frac{\partial^2 K^{(n)}(\Delta, \hat{u})}{\partial u^2}\right)^{-1/2} - \frac{1}{\hat{w}} + O((w - \hat{w})) \tag{36}$$

Thus, we have for the leading term of the approximation

$$\mathbb{P}(L_\Delta > a|Z_0 = z) = 1 - F_0(a|z) + f_0(a|z) \left\{ \frac{1}{\hat{u}}(K_0''(\hat{w}))^{1/2} \left(\frac{\partial^2 K^{(n)}(\Delta, \hat{u})}{\partial u^2}\right)^{-1/2} - \frac{1}{\hat{w}} \right\} \tag{37}$$

where $f_0(a|z)$ is the density function, and finally the first order saddlepoint approximation for the CDF is given by

$$\mathbb{P}(L_\Delta \leq a|Z_0 = z) \approx F_0(a|z) - f_0(a|z) \left\{ \frac{1}{\hat{u}}(K_0''(\hat{w}))^{1/2} \left(\frac{\partial^2 K^{(n)}(\Delta, \hat{u})}{\partial u^2}\right)^{-1/2} - \frac{1}{\hat{w}} \right\} \tag{38}$$

Higher order terms can be computed by similarly modifying the formulas of Lugannani and Rice [1980].

6 Error Analysis

In this section, we study the error incurred by the analytical approximation. Our overall approximation is a two stage process. In the first step, we have a short term expansion for the CGF of order-n_2, and in the second step, we have an order-n_1 saddlepoint approximation. Denote by $p^{(n_1,n_2)}$ the density function obtained as a result of this process.

We consider a saddlepoint approximation for a jump-diffusion process Z, corresponding to the portfolio loss (refer Proposition 4.2)

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t + dJ_t \tag{39}$$

where to simplify exposition and avoid the use of tensor notation, we consider the univariate case. In Equation (39), J_t is a jump process with intensity $\Lambda^*(t)$ and jump size distributions ν^* under $\mathbb{P}$. The characteristic function of the jump size is denoted by $\theta(\mathbf{i}u)$.

For the case of a pure diffusion process in Equation (39), the authors in Ait-Sahalia and Yu [2006] (Theorem 3), prove that the overall order of the two step process – CGF expansion and saddlepoint approximation – which uses a Gaussian leading term for the saddlepoint expansion is

$$p(\Delta, y|z) = p^{(n_1,n_2)}(\Delta, y|z)(1 + O(\Delta^{min(n_1+1,n_2/2)})) \tag{40}$$

where $p(\Delta, y|x)$ is the actual density. We thus see that the CGF expansion is the determining factor for the performance of this two stage approximation. While this result leads to useful insights for the simple case of a diffusion process, we need to address two issues in our case: a) the Markov process associated with our portfolio loss is a jump-diffusion process and not a pure diffusion process, and b) we use a non-Gaussian leading term for the saddlepoint approximation in lieu of a Gaussian leading term. In this section, we address both these issues and derive the modified error bounds for the density in the jump-diffusion case.

There have been previous works focusing on the error analysis of small-time expansions of jump processes. Ishikawa [1994] provides a small-time estimate of the density of a pure jump Lévy process. In Figueroa-López and Houdré [2009], the authors obtain small-time polynomial expansions for the tails of the loss distribution, assuming smoothness conditions on the Lévy density away from the origin. They also estimate the error in the expansion under additional regularity conditions. They show how the error term can be computed in terms of the Lévy density. Most of the previous papers handle expansions for the tail distribution of a Lévy process. In contrast our method is suitable for more generic jump diffusion processes. We focus on accurate estimates of the entire portfolio loss distribution instead of just dealing with the tails of a Lévy process. However, as widely done in current literature, we express the error of our approximation method in terms of the intensity $\Lambda^*(t)$ and the characteristic function $\theta(\mathbf{i}u)$ of the jump component.

Let $q^{(n_2)}(\Delta, y|z)$ denote the density corresponding to the approximate (order n_2 expansion) Laplace transform $\psi^{(n_2)}(\Delta, u|z)$

$$q^{(n_2)}(\Delta, y|z) = \mathbf{Re}\left[\frac{1}{2\pi}\int_u e^{-\mathbf{i}uy}\psi^{(n_2)}(\Delta, \mathbf{i}u|z)du\right] \tag{41}$$

We take the real part, as we cannot guarantee that the Fourier inverse of an approximate Laplace transform is real. We now analyze the two steps involved in the approximation

6.1 Analysis of the CGF approximation step

For the SDE specified in Equation (39), the expansion for the order-n_2 CGF and the Laplace transform can be written as

$$K^{(n_2)}(\Delta, u|z) = uz + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta + \frac{1}{2}\frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta^2 + \cdots + o(\Delta^{n_2})$$

and

$$\begin{aligned} \phi^{(n_2)}(\Delta, u|z) = \exp&\left(uz + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta \right) \\ &\times \left[1 + \frac{1}{2}\frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta^2 + \cdots + o(\Delta^{n_2}) \right] \end{aligned} \tag{42}$$

where the derivative $\frac{\partial^k}{\partial \Delta^k} K(\Delta, u|z)\Big|_{\Delta=0}$ is a polynomial of order $k+1$ in u for the case of a diffusion process, and a polynomial combined with the parameters of the jump component for a jump diffusion process Z. For example, we have for the second order derivative

$$\begin{aligned} \frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z) = {} & [\mu(\Delta, z)\mu'(\Delta, z) + \mu''(\Delta, z) - \mu(\Delta, z)]u \\ & + \left[\frac{1}{2}\mu(\Delta, z)(\sigma^2)'(\Delta, z) + \sigma^2(\Delta, z)\mu'(\Delta, z) + \frac{1}{2}(\sigma^2)''(\Delta, z) \right. \\ & \left. - \frac{1}{2}\sigma^2(\Delta, z) \right] u^2 + \frac{1}{2}\sigma^2(\Delta, z)(\sigma^2)'(\Delta, z)u^2 \\ & - \lambda^*(\theta(u) - 1)\left(\mu(\Delta, z)u + \frac{1}{2}\sigma^2(\Delta, z)u^2 \right) \\ & + \lambda^* \int e^{uj} \left[\mu(\Delta, z + j)u + \frac{1}{2}\sigma^2(\Delta, z + j)u^2 \right] \nu^*(dj) \end{aligned} \tag{43}$$

where $\lambda^* = \Lambda^*(0) = z_\Lambda$. To simplify computation, we collect the terms involving the jump component parameters in Equation (42) into a function $f(\lambda^*, \theta(\mathbf{i}u), u)$ and rewrite the equation as

$$\begin{aligned} \phi^{(n_2)}(\Delta, u|z) = \exp&\left(uz + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta \right) \\ &\times \left[\sum_{k=0}^{n_2} \sum_{j=0}^{k} \alpha_{k,j} u^j + f(\lambda^*, \theta(\mathbf{i}u), u) + o(\Delta^{n_2}) \right] \end{aligned} \tag{44}$$

where $\alpha_{k,j}$ depends only on z, through $\mu(\Delta, z), \sigma(\Delta, z)$ and their derivatives. We note that once the jump-diffusion process Z is completely specified, the terms in Equation (44) can be written exactly. We now have the following result

Proposition 6.1 *For an order-n_2 expansion of the CGF of a jump-diffusion process, the density corresponding to the approximate Laplace transform $\psi^{(n_2)}(\Delta, u|x)$ is given by*

$$q^{(n_2)}(\Delta, y|x) = \frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k \alpha_{k,j} O(\Delta^{-(j+1)/2})\right\} \quad (45)$$
$$+ \frac{1}{2\pi}\left\{\int_u \cos(u(z\Delta^{1/2} - \alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}\mathbf{Re}[f(\lambda^*, \theta(\mathbf{i}u), u)]du\right\}$$
$$+ \frac{1}{2\pi}\left\{\int_u \sin(u(z\Delta^{1/2} - \alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}\mathbf{Im}[f(\lambda^*, \theta(\mathbf{i}u), u)]du\right\}$$

Proof Proof in Appendix E.

6.2 Analysis of the saddlepoint approximation step

We now analyze the error incurred by the saddlepoint approximation in lieu of Fourier inversion. The non-Gaussian leading term (K_0) for the saddlepoint approximation for jump diffusion processes corresponds to the Laplace transform given by

$$\exp(K_0(\Delta, u|z)) = \exp(uz)\left[(1-\lambda^*\Delta)\exp\left(\mu(\Delta, z)\Delta u + \frac{1}{2}\sigma^2(\Delta, z)\Delta u^2\right) + \lambda^*\Delta\theta(u)\right]$$

To accommodate the new leading term, we use a new cumulant generating function K_0 which replaces the function $w^2/2$. Let f_0 denote the explicit density function corresponding to K_0. The saddlepoint approximation using the non-Gaussian leading term K_0 and with the order-n_2 approximation to the CGF is given by

$$p^{(0,n_2)}(\Delta, y|z) = f_0(y)\exp\left(\left\{K^{(n_2)}(\Delta, \hat{u}|z) - \hat{u}y\right\} - \left\{K_0(\hat{w}) - \hat{w}y\right\}\right)$$
$$\times (K_0''(\hat{w}))^{1/2}\left(\frac{\partial^2 K^{(n_2)}(\Delta, \hat{u}|z)}{\partial u \partial u^T}\right)^{-1/2} \quad (46)$$

where the saddlepoint $\hat{u}$ and $\hat{w}$ can be solved from

$$y = \frac{\partial}{\partial u}K^{(n_2)}(\Delta, \hat{u}|z)$$

and

$$y = \frac{\partial}{\partial u}K_0(\Delta, \hat{w}|z)$$

Different saddlepoint expansions with a non-Gaussian base can be obtained based on the function K_0, which is analytic at 0. The approximation to the density function $q^{(n_2)}(\Delta, y|z)$ can be written as

$$q^{(n_2)}(\Delta, z + x\Delta^{1/2}|Z_0 = z) = Re\left[\frac{1}{2\pi}\int_{\hat{u}-\mathbf{i}\infty}^{\hat{u}+\mathbf{i}\infty} e^{K^{(n_2)}(u) - u(z + x\Delta^{1/2})}du\right]$$

$$= Re\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{K^{(n_2)}(\hat{u}+\mathbf{i}w)-(\hat{u}+\mathbf{i}w)(z+x\Delta^{1/2})}dw\right]$$

For a non-Gaussian base, consider the first and second order approximations to the cumulant generating function. We have

$$K^{(1)}(\Delta,u|z) = uz + \Delta\left[\mu(\Delta,z)u + \frac{1}{2}\sigma(\Delta,z)^2u^2 + \lambda^*(\theta(u)-1)\right]$$

and

$$K^{(2)}(\Delta,u|z) = uz + \Delta\left[\mu(\Delta,z)u + \frac{1}{2}\sigma(\Delta,z)^2u^2 + \lambda(\theta(u)-1)\right] + \log\left[1 + \frac{1}{2}\Delta^2\frac{\partial^2}{\partial\Delta^2}K(\Delta,u|x)\bigg|_{\Delta=0}\right] \quad (47)$$

where the derivative $\frac{\partial^2}{\partial\Delta^2}K(\Delta,u|z)$ is as given in Equation (43). The order-n_2 approximation to the CGF $K^{(n_2)}(\Delta,u|z)$ combines a polynomial in u whose coefficients are dependent on the values of $\mu(\Delta,z)$ and $\sigma(\Delta,z)$ and their derivatives, with a function $g(\lambda^*,\theta(u),u)$ dependent on the jump components λ^* and $\theta(\mathbf{i}u)$. In order to simplify exposition we decompose $K^{(n_2)}(\Delta,u|z)$ as

$$K^{(n_2)}(\Delta,u|z) = \sum_{k=0}^{n_2}\sum_{j=0}^{k}\alpha_k u^j + \ln[g(\lambda^*,\theta(u),u)] + O(\Delta^{n_2}) \quad (48)$$

denoting the contribution from the polynomial and jump components. As in Section 6.1, we have that the terms in Equation (48) can be exactly written, once the jump-diffusion dynamics have been precisely specified. We have the following proposition

Proposition 6.2 *For an order-n_2 expansion of the CGF of a jump-diffusion process, the error induced due to the order-n_1 non-Gaussian base saddlepoint approximation is given by*

$$p^{(n_1,n_2)}(\Delta,x+z\Delta^{1/2}|x) = q^{(n_2)}(\Delta,x+z\Delta^{1/2}|x)(1+O(\Delta^{n_1+1}))\times$$
$$\mathbf{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\mathbf{i}w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta)-\frac{1}{2}\sigma^2(x)\Delta w^2\right)\times \frac{g(\lambda^*,\theta(\hat{u}+\mathbf{i}w),\hat{u}+\mathbf{i}w)}{g(\lambda^*,\theta(\hat{u}),\hat{u})}dw\right]$$

Proof The proof, which uses Proposition 6.1 is given in Appendix F.

6.3 Using the Error Bounds

Propositions 6.1 and 6.2 derive expressions for the approximate density $q^{(n_2)}(\Delta,y|x)$ in terms of $\mu(\Delta,z),\sigma(\Delta,z),\lambda^*$ and $\theta(\mathbf{i}u)$, for both the steps in our approximation. We can also conclude from the above propositions that for an SDE as in Equation (39), the explicit error bounds of each of the two steps in our approximation can be computed once we have the jump component specification. This is due to the fact that specifying the jump component allows us to compute the functions $f(\lambda^*, \theta(\hat{u}), \hat{u})$ and $g(\lambda^*, \theta(\hat{u}), \hat{u})$. With the functions f and g specified, we can evaluate the integrals in the propositions to compute the order of our approximation error. In this way, the expressions we derive offer useful bounds for the error in each of the steps in our two step approximation. As an example of this procedure, in Appendix G we specify the jump dynamics of the process Z and illustrate the computation of the error bound for the Poisson default intensity case.

7 Handling Longer Time Horizons

In this section, we analyze techniques to better approximate the portfolio loss distribution at longer time horizons. There are two different approaches to this problem. The first is to include higher order terms in the CGF expansion. We show how higher order analytical approximations for the terms of the CGF can be attained by ignoring very high order terms which contribute very little to the distribution. This allows us to ignore a few of the higher order terms, which can be tedious to compute. As an alternative to manual computation, we derive a recursive equation for the coefficients of the higher order CGF terms. This method leads itself to computation of higher order terms by symbolic derivatives.

The second method to improve the long-time approximation is to consider alternative methods to compute the CGF. In Section 9 we show that the small-time approximation method yields good estimates for the distribution of portfolio loss for maturities ranging from 3 to 6 months. In order to improve the performance beyond these maturities, we investigate the effect of combining several smaller time horizon approximations in Section 7.3.

7.1 Higher Order Analytical Approximations

From Equation (23), we have the second order expansion to the CGF of the portfolio loss. For relatively large portfolios which are well diversified, this expansion is a good approximation to the portfolio loss distribution (see the results in Section 9). In this section, we modify the approximation by considering higher order terms which improve the performance of the CGF for longer time horizons and highly correlated portfolios.

The n^{th} order approximation to the CGF is given by:

$$K^{(n)}(\Delta, u|z) = uz_l + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta + \frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta^2 + \cdots + O(\Delta^{n+1})$$

From Equation (17), applying the Taylor expansion on $log(\psi(H, u|z))$, we the following higher order terms for the CGF:

$$K^{(1)}(\Delta, u|z) = uz_L + c_1 \Delta$$

$$K^{(2)}(\Delta,u|z) = uz_L + c_1\Delta + \left(\frac{c_1^2}{2} - c_2\right)\Delta^2$$

$$K^{(3)}(\Delta,u|z) = uz_L + c_1\Delta + \left(\frac{c_1^2}{2} - c_2\right)\Delta^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{3}\right)\Delta^3$$

$$K^{(4)}(\Delta,u|z) = K^{(3)}(\Delta,u|z) + \left(c_4 - \frac{c_2^2}{2} - c_1c_3 + c_1^2c_2\right)\Delta^4 \tag{49}$$

where c_1 and c_2 are as defined in Equation (75-76). The higher order correction terms, c_3 and c_4 are defined as:

$$c_3 = \frac{1}{3}\frac{\mathcal{A}^3 e^{uz_l}}{e^{uz_l}} \tag{50}$$

$$c_4 = \frac{1}{4}\frac{\mathcal{A}^4 e^{uz_l}}{e^{uz_l}} \tag{51}$$

In order to make the expansion tractable, and to ensure that we only keep the terms which eventually contribute to a large fraction of the higher order terms, we propose to drop some of the terms involving the parameters ι, ϵ and ξ. From the expansion of c_2 (refer equation 76), we see that some of the expressions involving these parameters are high-order product terms, with each element of the product less than unity, and several terms which vanish if many reference names default. This corresponds to a high value of the loss function, which is the region where we want to improve the expansion. Hence, since in our region of interest some of the terms involving these parameters contribute little to the expansion, and we ignore them in the higher order expansion terms (c_3 and c_4). We thus have:

$$\begin{aligned}
3c_3 \approx{}& \sum_j \lambda_j(1-n_j)u^2 \sum_k \sum_q g_t^{j,k} g_t^{j,q} e^{u\sum_m g^{j,m}} \\
&+ \sum_j \lambda_j(1-n_j)u \sum_k g_{tt}^{j,k} e^{u\sum_m g^{j,m}} \\
&+ \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k g^{j,k}} - 1\right)\right]_z \mu_t \\
&+ \left[\sum_j \lambda_j(1-n_j)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_z \mu \\
&+ \frac{1}{2}tr\left[\left[\sum_j \lambda_j(1-n_j)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_{zz} \sigma\sigma^T\right] \\
&+ \sum_m \iota_m \left(\sum_j \epsilon(1-n_j)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right)
\end{aligned}$$

$$+ \sum_j \lambda_j(1-n_j)\left[\left(\sum_l (\lambda_l+\xi_{l,j})(1-n_l)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right) e^{u\sum_k g^{j,k}}\right]$$

$$+ \sum_j \lambda_j(1-n_j)\left[\left(\sum_l (\lambda_l+\xi_{l,j})(1-n_l)\left(e^{u\sum_k g^{j,k}}-1\right)\right)\left(u\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]$$

$$- \sum_j \lambda_j(1-n_j)\left(\sum_i \lambda_i(1-n_i)u\sum_k g_t^{i,k} e^{u\sum_m g^{i,m}}\right)$$

$$+ \left[\sum_j \lambda_j(1-n_j)u^2\sum_k\sum_q g_t^{j,k} g_t^{j,q} e^{u\sum_m g^{j,m}}\right]_z \mu$$

$$+ \left[\sum_j \lambda_j(1-n_j)u\sum_k g_{tt}^{j,k} e^{u\sum_m g^{j,m}}\right]_z \mu$$

$$+ \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k g^{j,k}}-1\right)\right]_{zz} \mu_t$$

$$+ \left[\sum_j \lambda_j(1-n_j)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_{zz} \mu$$

$$- \left[\sum_j \lambda_j(1-n_j)\left(\sum_i \lambda_i(1-n_i)u\sum_k g_t^{i,k} e^{u\sum_m g^{i,m}}\right)\right]_z \mu \tag{52}$$

where we have dropped the second order derivatives involving the σ, ϵ, ξ and ι parameters. In the expansion of c_4, we only preserve the time derivatives in order to get a tractable expansion. We preserve terms involving u as a multiplying factor, as at higher losses these terms dominate those with several multiplicative factors without u as one of terms. We have:

$$4c_4 \approx \sum_j \lambda_j(1-n_j)u^3\sum_k\sum_q\sum_r g_t^{j,k} g_t^{j,q} g_t^{j,r} e^{u\sum_m g^{j,m}}$$

$$+ \sum_j \lambda_j(1-n_j)u^2\sum_k\sum_q (g_t^{j,k} g_{tt}^{j,q} + g_{tt}^{j,k} g_t^{j,q}) e^{u\sum_m g^{j,m}}$$

$$+ \sum_j \lambda_j(1-n_j)u^2\sum_k\sum_r g^{j,r} g_{tt}^{j,k} e^{u\sum_m g^{j,m}}$$

$$+ \sum_j \lambda_j(1-n_j)u\sum_k g_{ttt}^{j,k} e^{u\sum_m g^{j,m}}$$

$$+ \left[\sum_j \lambda_j(1-n_j)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_z \mu_t$$

$$
\begin{aligned}
&+ \left[\sum_j \lambda_j(1-n_j)u^2\left(\sum_k\sum_r g_t^{j,k} g_t^{j,r} e^{u\sum_m g^{j,m}}\right)\right]_z \mu \\
&+ \frac{1}{2}tr\left[\left[\sum_j \lambda_j(1-n_j)u^2\left(\sum_k\sum_r g_t^{j,r} g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_{zz} \sigma\sigma^T\right] \\
&+ \sum_m \iota_m\left(\sum_j \epsilon(1-n_j)u\left(\sum_k\sum_r g_t^{j,r} g_t^{j,k} e^{u\sum_m \delta_m g^{j,m}}\right)\right) \\
&- \sum_j \lambda_j(1-n_j)\left(\sum_i \lambda_i(1-n_i)u^2\sum_k\sum_r g_t^{i,k} g^{i,r} e^{u\sum_m g^{i,m}}\right) \\
&+ \left[\sum_j \lambda_j(1-n_j)u^2\sum_k\sum_r g_t^{j,r} g_{tt}^{j,k} e^{u\sum_m g^{j,m}}\right]_z \mu \\
&+ \left[\sum_j \lambda_j(1-n_j)u\left(\sum_k g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_{zz} \mu_t \\
&+ \left[\sum_j \lambda_j(1-n_j)u^2\left(\sum_k\sum_r g_t^{j,r} g_t^{j,k} e^{u\sum_m g^{j,m}}\right)\right]_{zz} \mu \\
&- \left[\sum_j \lambda_j(1-n_j)\left(\sum_i \lambda_i(1-n_i)u^2\sum_k\sum_r g_t^{i,r} g_t^{i,k} e^{u\sum_m g^{i,m}}\right)\right]_z \mu \qquad (53)
\end{aligned}
$$

Appendix D has details regarding the various partial derivatives involved in Equations (52)-(53). In Section 9, we study the improvement achieved when we include these higher order approximations.

7.2 Approximation using Symbolic Derivatives

In this section, we study the use of symbolic derivatives in computing higher order terms of the expansion. The presence of terms which are repeated often in the higher order expansions motivates this approach. We start by introducing a new term $\Psi(j)$ defined as

$$
\Psi(j) = \left(e^{u\sum_k g^{j,k}} - 1\right) \qquad (54)
$$

Using this term, the expressions for c_1 and c_2 simplify to

$$
c_1 = \sum_j \lambda_j(1-n_j)\Psi(j) \qquad (55)
$$

and

$$
\begin{aligned}
2c_2 = &\sum_j \lambda_j(1-n_j)\Psi_t(j) + \left[\sum_j \lambda_j(1-n_j)\Psi(j)\right]_z \mu \\
&+ \frac{1}{2}tr\left[\left[\sum_j \lambda_j(1-n_j)\Psi(j)\right]_{zz}\sigma\sigma^T\right] + \sum_m \iota_m\left(\sum_j \epsilon(1-n_j)\Psi(j)\right) \\
&+ \sum_j \lambda_j(1-n_j)\left[\left(\sum_l(\lambda_l+\xi_{l,j})(1-n_l)\Psi(l)\right)(\Psi(j)+1) - c_1\right]
\end{aligned} \tag{56}
$$

which is equal to

$$
\begin{aligned}
2c_2 = &\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial z}\mu + \frac{1}{2}tr\left[\frac{\partial^2 c_1}{\partial z^2}\sigma\sigma^T\right] + \sum_m \iota_m\left(\sum_j \epsilon(1-n_j)\Psi(j)\right) \\
&+ c_1^2 + \sum_j \lambda_j(1-n_j)\left(\sum_l \xi_{l,j}(1-n_l)\Psi(l)\right)(\Psi(j)+1)
\end{aligned} \tag{57}
$$

With the expressions for c_1 and c_2 re-written as in Equations [55-57], the higher order expansions can be evaluated as derivatives of lower order terms when the infinitesimal generator is applied. This leads itself to an efficient representation for computation by symbolic derivatives. The recursive expression for c_n can be written as

$$
\begin{aligned}
nc_n &= \frac{1}{(n-1)!}\frac{\mathcal{A}^n e^{uz_l}}{e^{uz_l}} = \frac{\mathcal{A}\left(c_{n-1}e^{uz_l}\right)}{e^{uz_l}} \\
&= \mathcal{A}c_{n-1} + c_{n-1}e^{-uz_l}\mathcal{A}e^{uz_l} \\
&= \mathcal{A}c_{n-1} + c_{n-1}c_1
\end{aligned} \tag{58}
$$

The symbolic derivative method, is thus an easy way to incorporate all the higher order product terms in the expansion. The only drawback of this method is the lack of interpretability of the results. The expansion obtained from this method does not lead itself to easy interpretation when compared to the analytical method. Section 9 compares the result of using the symbolic derivative method in lieu of the analytical approximation for higher order expansions of the CGF.

7.3 Composition of Short Time Expansions

In this Section, we study the effect of combining several small time approximations to handle longer maturities. The main idea is explained next. Given the Markov process M for the portfolio loss, we use the tower property of conditional expectation on the order-2 Laplace transform and obtain

$$
\psi(\Delta_2, u|Z_0 = z_0) = \mathbb{E}(e^{uL_{\Delta_2}}|Z_0 = z_0)
$$

$$
\begin{aligned}
&= \mathbb{E}\left(\mathbb{E}(e^{uL_{\Delta_2}}|Z_{\Delta_1} = z_1)|Z_0 = z_0\right) \\
&= \mathbb{E}(\psi^*(\Delta_2 - \Delta_1, u, z_1)|Z_0 = z_0)
\end{aligned} \tag{59}
$$

where $\Delta_2 \geq \Delta_1$ and $\psi^*(\Delta_2 - \Delta_1, u, z_1) = \mathbb{E}(e^{uL_{\Delta_2}}|Z_{\Delta_1} = z_1)$. We can thus express the approximate Laplace transform at a maturity Δ_2 by combining two smaller time order-2 approximations of maturity $\Delta_2 - \Delta_1$ and Δ_1 respectively. Note that this results in a different expression than the higher (> 2) order expansion for the Δ_2 maturity. This method is motivated by the fact that the small-time approximation is well suited for short time horizons and easy to compute to the first few orders. Instead of tackling a higher order expansion of the longer time horizon δ_2, it may be beneficial to compose several smaller order approximations, each of a smaller maturity. The approximation for ψ^* is as explained in Section 4. We handle the outer expansion in what follows. From Equations (17) and (59) we have

$$
\begin{aligned}
\psi(\Delta_2, u|Z_0 = z_0) &= \mathbb{E}(\psi^*(\Delta_2 - \Delta_1, u, z_1)|Z_0 = z_0) \\
&= \mathbb{E}\left(\sum_{m=0}^{M} \frac{(\Delta_2 - \Delta_1)^m}{m!} \mathcal{A}^m e^{u(z_1)_L}|Z_0 = z_0\right)
\end{aligned} \tag{60}
$$

As shown in Appendix A, the function e^{uL_Δ} and all its derivatives are bounded and continuous on a compact support, and hence the terms inside the summation in Equation (60) are in the domain $\mathcal{D}_\mathcal{A}$ of the infinitesimal generator $\mathcal{A}$. Performing a second short-term expansion we have

$$
\begin{aligned}
\psi(\Delta_2, u|Z_0 = z_0) &\approx \mathbb{E}\left(\sum_{m=0}^{M} \frac{(\Delta_2 - \Delta_1)^m}{m!} \mathcal{A}^m e^{u(z_1)_L}|Z_0 = z_0\right) \\
&= \sum_{n=0}^{N} \frac{(\Delta_1)^n}{n!} \mathcal{A}^n \left(\sum_{m=0}^{M} \frac{(\Delta_2 - \Delta_1)^m}{m!} \mathcal{A}^m e^{u(z_0)_L}\right) \\
&= \sum_{n=0}^{N} \frac{(\Delta_1)^n}{n!} \mathcal{A}^n \left[e^{u(z_0)_L}\left(1 + c_1(\Delta_2 - \Delta_1) + c_2(\Delta_2 - \Delta_1)^2\right)\right] \\
&= e^{u(z_0)_L}\left(1 + c_1(\Delta_2 - \Delta_1) + \frac{c_2}{2}(\Delta_2 - \Delta_1)^2\right) \\
&+ \Delta_1 e^{u(z_0)_L}\left(c_1 + c_2(\Delta_2 - \Delta_1) + \frac{c_3}{2}(\Delta_2 - \Delta_1)^2\right) \\
&+ \frac{\Delta_1^2}{2} e^{u(z_0)_L}\left(c_2 + c_3(\Delta_2 - \Delta_1) + \frac{c_4}{2}(\Delta_2 - \Delta_1)^2\right) \\
&= e^{u(z_0)_L}\left(1 + c_1\Delta_2 + \frac{c_2}{2}\Delta_2^2 + \frac{c_3}{2}\Delta_1\Delta_2(\Delta_2 - \Delta_1)\right. \\
&\qquad \left. + \frac{c_4}{4}\Delta_1^2(\Delta_2 - \Delta_1)^2\right)
\end{aligned} \tag{61}
$$

for the order 2 approximation of the long term Laplace transform $\psi(\Delta_2, u|Z_0 = z_0)$. We note that Equation (61) agrees with the usual short-term approximation as in Equation

(17) up to order-2 in Δ_2. If $\Delta_1 = 0$ or $\Delta_1 = \Delta_2$, we again get the usual expansion for the maturity Δ_2. Rewriting the equation we have

$$\psi(\Delta_2, u|Z_0 = z_0) = e^{u(z_0)_L}\left(1 + c_1\Delta_2 + \frac{c_2}{2}\Delta_2^2 + \frac{c_3}{6}\Delta_2^3 + \frac{c_4}{24}\Delta_2^4\right) - \frac{c_3}{6}((\Delta_2 - \Delta_1)^3 + \Delta_1^3) - \frac{c_4}{24}((\Delta_2 - \Delta_1)^4 + \Delta_1^4) \quad (62)$$

which is the expression for the order-4 expansion of the Laplace transform at a maturity Δ_2 with a few correction terms. We thus note that the compound expansion only introduces error terms in the correct expansion of the Laplace transform. The reason for this is as follows. As we note from Equation (61), while calculating the compound expansion, we still need to compute higher order coefficients c_3 and c_4 which are independent of the time horizon. However, the compound expansion ignores the higher order terms in Δ_2 and Δ_1, as seen from Equation (62).

8 Examples

In Section 4.3, we considered the case of a generic jump diffusion intensity specification and derived expressions for the CGF approximation. We now consider specific cases of the jump diffusion intensity dynamics and show how the terms of the CGF can be approximated. We study the case of doubly-stochastic and self-exciting intensity models.

8.1 Doubly-Stochastic Model

The doubly-stochastic model for firm intensities is quite widely used. Similar models are used in Chava and Jarrow [2004], Duffie and Garleanu [2001], Eckner [2008], Eckner [2009], Feldhutter [2007], Papageorgiou and Sircar [2007] and others. In this formulation, the actual default intensity of a reference name $j \in \{1, 2, \ldots, n\}$, is specified by Λ_j, a function of the firm specific idiosyncratic factor X_j and several systematic factors $X_{n+k}, k \in \{1, 2, \ldots, K\}$. Conditional on the realization of these factors, the default time of reference name j is the first jump time of an inhomogeneous Poisson process with conditionally deterministic intensity $\lambda_j = \Lambda_j(X_j, X_{n+1}, X_{n+2}, \ldots, X_{n+K})$. The idiosyncratic and the systematic factors are independent of each other and conditional on the realization of the K systematic factors, the default times of each reference name are $\mathbb{P}$-independent of each other.

For modeling purposes, we assume that the risk-factors defined above follow $\mathbb{P}$-feller diffusions. The dynamics are specified as

$$dX_m(t) = \kappa_{X_m}(\theta_{X_m} - X_m(t))dt + \sigma_{X_m}\sqrt{X_m(t)}dW_m(t), \quad m = 1, 2, \ldots, n + K$$

where the W_ms are $\mathbb{P}$-independent Brownian motions. To ensure non-negativity a.s. we assume $2\kappa_{X_m}\theta_{X_m} > \sigma_{X_m}^2$. The default intensities of each reference name are specified by

$$\lambda_j(t) = X_j(t) + \sum_{k=0}^{K} \omega_j^k X_{n+k}(t) \quad (63)$$

n structure between the reference names. Duffie and Garleanu [2001] derive restrictions that must be satisfied for the firm intensity λ_j as define above, to follow a Feller diffusion. For the Feller diffusion triplet of risk-factor m, denoted by $(\kappa_{X_m}, \theta_{X_m}, \sigma_{X_m})$, we must have

$$\kappa_{X_m} = \kappa, m = 1, 2, \ldots, n + K$$

$$\sigma_{X_j} = \sqrt{\omega_j^k}\sigma_{n+k}, k = 1, 2, \ldots, K, \forall j = 1, 2, \ldots, n$$

where the ω_j^k are parameters that control the correlatioWe thus have the following for the $\mathbb{P}$-Feller diffusion triplet for the reference name default intensity λ_j

$$(\kappa_j, \theta_j, \sigma_j) = (\kappa, \theta_{X_j} + \sum_{k=1}^{K} \omega_j^k \theta_{X_{n+k}}, \sigma_{X_j}) \tag{64}$$

Given the intensity specification in terms of X_t, the augmented vector Z_t in this case is thus given by $Z_t = (X_t, N_t, L_t)$. For the specified dynamics of the factors and intensities, the drift vector for the augmented Markov process (t, Z_t) can be written as

$$\begin{pmatrix} \mu^{x_i} \\ \mu^{n_i} \\ \mu^{l} \end{pmatrix} = \begin{pmatrix} \kappa(\theta_{X_i} - X_i(t)) \\ 0 \\ 0 \end{pmatrix} \tag{65}$$

and the volatility matrix is given by

$$\begin{pmatrix} \sigma^{x_i} \\ \sigma^{n_i} \\ \sigma^{l} \end{pmatrix} = \begin{pmatrix} \mathbf{e_i}.\sigma_{X_i}\sqrt{X_i(t)} \\ 0 \\ 0 \end{pmatrix} \tag{66}$$

Let $g^j(t, X_t, L_t) = l_j$. The X_t components do not have jumps while N_t has unit jumps. The loss component L_t has jumps of size $\sum_k g^{jk}(t, X_t, L_t)$ for each jump in N_t^j. We get the following for the expansions of the CGF

$$c_1 = \sum_j \lambda_j (1 - n_j) \left(e^{u \sum_k g^{jk}} - 1 \right) \tag{67}$$

and

$$\begin{aligned} 2c_2 = & \sum_j \lambda_j (1 - n_j) u \sum_k g_t^{j,k} e^{u \sum_m g^{j,m}} \\ & + \left[\sum_j \lambda_j (1 - n_j) \left(e^{u \sum_k g^{j,k}} - 1 \right) \right]_z \mu \\ & + \frac{1}{2} tr \left[\left[\sum_j \lambda_j (1 - n_j) \left(e^{u \sum_k g^{j,k}} - 1 \right) \right]_{zz} \sigma \sigma^T \right] \\ & + \sum_j \lambda_j (1 - n_j) \left[\left(\sum_l \lambda_l \left(e^{u \sum_k g^{j,k}} - 1 \right) \right) e^{u \sum_k g^{j,k}} - c_1 \right] \end{aligned} \tag{68}$$

8.2 Self-Exciting Model

The correlation structure in the doubly-stochastic model comes from the dependence of the reference name's intensity on the systematic factors. Another method of modeling correlation is the self-exciting framework, in which defaults are correlated as every default has an impact on the intensity of the surviving firms. Self-exciting models are considered in Errais et al. [2010], Jarrow and Yu [2001] and others. The default intensity of reference name j follows

$$d\lambda_j(t) = \kappa_j(\theta_j - \lambda_j(t))dt + \sum_o \xi_{j,o} dN_t^o \tag{69}$$

where κ_j and θ_j are the mean-reversion speed and level parameters respectively, and $\xi_{j,o}$ models the impact of firm o's default on the intensity of firm j. We model the short rate and liquidity processes as in the previous section. The drift vector for the augmented Markov process $Z_t = (\Lambda_t, N_t, L_t)$ in this case can be written as

$$\begin{pmatrix} \mu^{\lambda_j} \\ \mu^{n_i} \\ \mu^l \end{pmatrix} = \begin{pmatrix} \kappa_j(\theta_j - \lambda_j(t)) \\ 0 \\ 0 \end{pmatrix} \tag{70}$$

and the volatility matrix is given by

$$\begin{pmatrix} \sigma^{\lambda_j} \\ \sigma^{n_i} \\ \sigma^l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{71}$$

Let $g^{jk}(t, \Lambda_t, L_t) = l_{jk}$. The X_t components do not have jumps while N_t has unit jumps. The loss component L_t has jumps of size $\sum_k g^{jk}(t, \Lambda_t, L_t)$ for each jump in N_t^j. We get the following for the expansions of the CGF

$$c_1 = \sum_j \lambda_j(1 - n_j)\left(e^{u\sum_k g^{j,k}} - 1\right) \tag{72}$$

and

$$\begin{aligned} 2c_2 = & \sum_j \lambda_j(1-n_j) u \sum_k g_t^{j,k} e^{u\sum_m g^{j,m}} \\ & + \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k g^{j,k}} - 1\right)\right]_z \mu \\ & + \sum_m \iota_m \left(\sum_j \epsilon(1-n_j)(e^{u\sum_k g^{j,k}} - 1)\right) \\ & + \sum_j \lambda_j(1-n_j)\left[\left(\sum_l (\lambda_l + \xi_{l,j})(1-n_l)\left(e^{u\sum_k g^{j,k}} - 1\right)\right) e^{u\sum_k g^{j,k}} - c_1\right] \end{aligned} \tag{73}$$

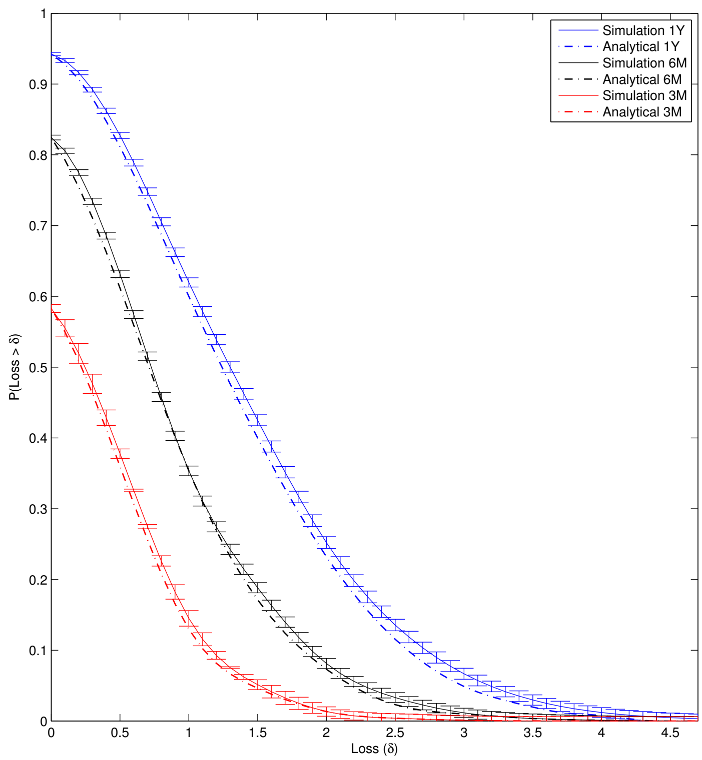

Figure 3: Inverse CDF comparing performance of Analytical Method and Simulations for the Type-I portfolio

9 Numerical Results

We test our analytical approach using an efficient Monte Carlo Algorithm (A/R scheme) presented in Giesecke, Kim, and Zhu [2010b]. We use the doubly stochastic model of default timing with one common factor to analyze the performance of the small-time approximation. We consider two portfolios, one consisting of high quality names with low correlations (Type I), and the other consisting of high risk names with higher correlations (Type II) to test the effectiveness of the various approximations. The parameters of the model are chosen as follows. In the Type I portfolio, for each constituent firm $i = 1, \dots, 100$, we draw κ^i from $U[0.5, 1.5]$, θ^i from $U(0.001, 0.051)$. We choose $\sigma^i = \min(\sqrt{\kappa^i \theta^i}, \bar{\sigma}^i)$, where $\bar{\sigma}^i = U[0, 0.2]$. We initialize the idiosyncratic risk factor X_0^i equal to its long-run mean θ^i. For the common risk factor we set $X_0^0 = \theta^0 = 0.01$, $\kappa^0 = 0.5$, $\sigma^0 = 0.2$. We set the loss given default at $l = 0.3$ for all assets.

In the Type II portfolio, for each constituent firm $i = 1, \dots, 100$, we draw κ^i from $U[0.8, 1.8]$, θ^i from $U(0.001, 0.079)$. We choose $\sigma^i = \min(\sqrt{\kappa^i \theta^i}, \bar{\sigma}^i)$, where $\bar{\sigma}^i = U[0.15, 0.4]$. We initialize the idiosyncratic risk factor X_0^i equal to its long-run mean θ^i. For the common risk factor we set $X_0^0 = \theta^0 = 0.07$, $\kappa^0 = 0.25$, $\sigma^0 = 0.5$. We choose the loss given default from $U[0.1, 0.8]$ for each asset. We construct two sets of portfolio

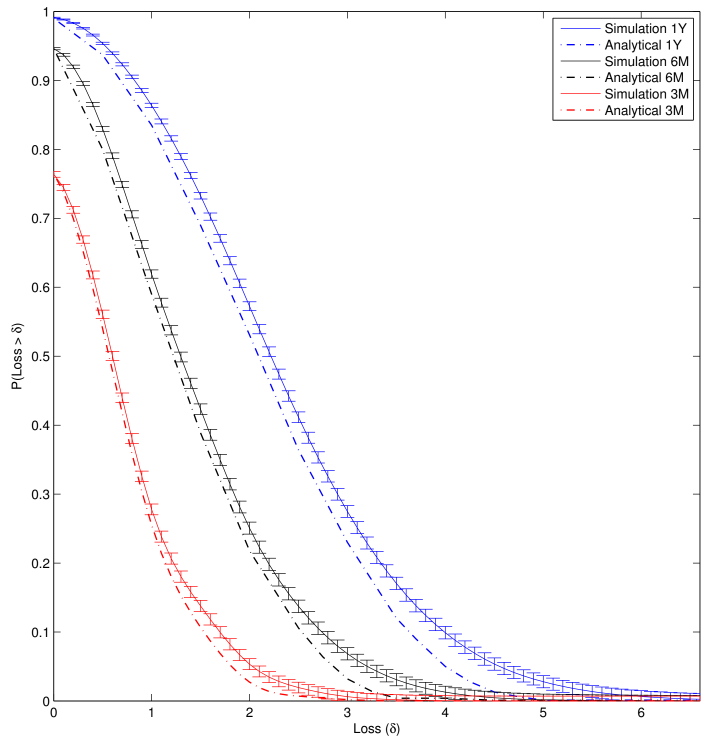

Figure 4: Inverse CDF comparing performance of Analytical Method and Simulations for the Type-II portfolio

weights that are distributed randomly among the available $m = 100$ names.

Tables [1] and [2] contains the running times for the A/R scheme described in Giesecke et al. [2010b] with an error threshold $\epsilon = 10^{-8}$. We note that we can provide a bound on the bias of this scheme using Proposition 3.2 of Giesecke et al. [2010b]. The experiments were performed on a desktop PC with a Core i7 4GhZ processor and 12GB of RAM running Linux Kernel (2.6.34). We use the GNU scientific library (GSL) for random number generation and evaluating special functions such as the Kummer confluent hypergeometric function. From the table we see that the traditional numerical schemes take considerable time in evaluating the loss distribution for a portfolio of $m = 100$ names. Tables [1] and [2] also contain the running times for the analytical method. We see that the analytical method is faster by several orders of magnitude, making this method suitable for large portfolios with thousands of names, and extends itself to very efficient scenario analysis. We also note that the simulation scheme has very small bias, which shows the accuracy of the simulation results, allowing us to use the efficient Monte Carlo simulation scheme as a good benchmark.

Figures (3) and (4) show the analytical and simulation results (order-2) for the loss distribution function of the portfolio loss. For a loss level δ, the figure plots the probability that a loss greater than δ occurs, for different time horizons. We also plot the 95% confidence intervals for the simulation scheme. We compare the performance of the small-time approximation on the two portfolios first. As the time horizon increases, the number of defaults increase leading to higher probability of loss of a particular level. The small-time approximation performs very well for the Type-I portfolio for time horizons up to 6 to 9 Months, with most values within the 95% confidence bounds of the A/R scheme. For the Type-II portfolio, the approximation degrades in performance in the tail of the distribution. Without the presence of higher order terms, the approximation has less curvature, leading to thinner tails in the case of the Type-II portfolio. This motivates our approach to consider higher order terms and alternative approximation methods to handle highly correlated portfolios and longer time horizons.

We now study the performance improvement when we include higher order terms in the small-time expansion. As shown in Section 7, the CGF approximation can be improved by incorporating higher order terms in the expansion. We expect these terms to be especially useful in the tail of the distribution. Figures [5] and [6] show the distributions obtained when we use the $K^{(4)}$ expansion of the CGF as in Equation (49). In the Type-I portfolio case, from Figure [5] we see that the improvement when using the higher order $K^{(4)}$ expansion is not significant. We note that the higher order expansion marginally outperforms the $K^{(2)}$ expansion in this case. However, in the case of the Type-II portfolio, Figure [6] shows that including the higher order terms does lead to significant improvement in the tails of the distribution. The $K^{(2)}$ expansion is not adequate to capture the features of the loss distribution, for highly correlated portfolios and for longer maturities.

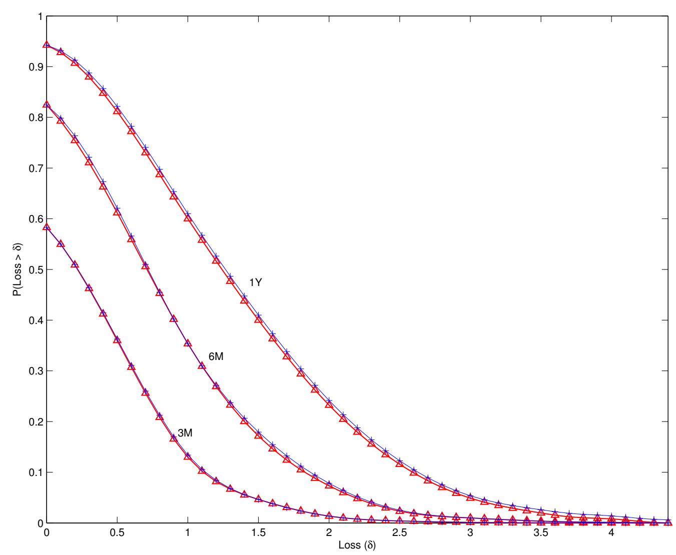

Figure 5: Inverse CDF comparison with and without higher order terms for the Type-I portfolio. The red curve is obtained using the $K^{(2)}$ approximation for the CGF and the blue curve using the $K^{(4)}$ approximation

Figure [7] shows the behavior of the difference in logarithm of the inverse CDF and the logarithm of the empirical distribution inverse CDF at a 6 month maturity, obtained by two different methods, a) using the CGF computed from a 3^{rd} order symbolic derivative

Table 1: Run times for the Type-I portfolio

Method	Maturity	Trials	Bias Bound	Time (sec)
Small-Time Approx (order 3)	3M	N/A	N/A	19.6
	6M	N/A	N/A	19.5
	1Y	N/A	N/A	19.6
Symbolic Approx (order 3)	3M	N/A	N/A	30.5
	6M	N/A	N/A	31.2
	1Y	N/A	N/A	35.8
Simulation	3M	500,000	2.18E-4	2645.8
	6M	500,000	4.05E-4	2795.4
	1Y	500,000	8.13E-4	2856.4

Table 2: Run times for the Type-II portfolio

Method	Maturity	Trials	Bias Bound	Time (sec)
Small-Time Approx (order 3)	3M	N/A	N/A	19.4
	6M	N/A	N/A	19.7
	1Y	N/A	N/A	19.5
Symbolic Approx (order 3)	3M	N/A	N/A	31.3
	6M	N/A	N/A	32.9
	1Y	N/A	N/A	38.3
Simulation	3M	500,000	1.16E-3	2743.6
	6M	500,000	2.38E-3	2918.4
	1Y	500,000	4.74E-3	3096.3

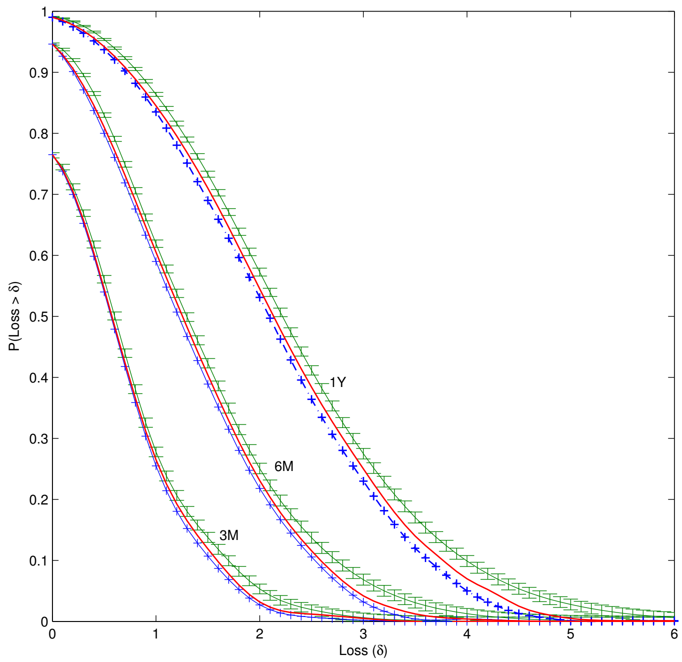

Figure 6: Inverse CDF comparison with and without higher order terms for the Type-II portfolio. The red curve is obtained using the $K^{(4)}$ approximation for the CGF and the blue curve using the $K^{(2)}$ approximation. The green curve shows the simulation results.

($K^{(3)}_{Symbolic}$) and b) using the 3^{rd} order analytical approximation ($K^{(3)}_{Analytical}$) as derived in Section 7.2. Unlike the analytical approximation, we can easily incorporate all the higher order product terms using the symbolic derivative expansion. We used the symbolic derivative package in MATLAB to perform the computations. From the figure we see that in the tails of the distribution, the symbolic derivative better approximates the CDF. In the body of the loss distribution we do not see a big difference in these two methods. Figure [8] shows the same analysis for a maturity of 1 year. We see a similar behavior in the body of the distribution with the difference in the tails being much smaller. At longer maturities, the inclusion of higher order terms in the symbolic derivative does to lead to much improvement in the approximation, and we can instead use the easier to interpret analytical approximation. Tables [1] an [2] contain the computation time of the symbolic approximation method for the order-3 expansions.

We checked all the above results by using numerical Fourier Analysis in lieu of the saddlepoint method, in order to check its accuracy. We confirmed that the resulting distributions are very similar as that produced by the saddlepoint method.

10 Conclusion

In this paper, we develop analytical approximations for the loss distribution of a loan portfolio, consisting of corporate loans, and other credit derivatives. The ability to analytically

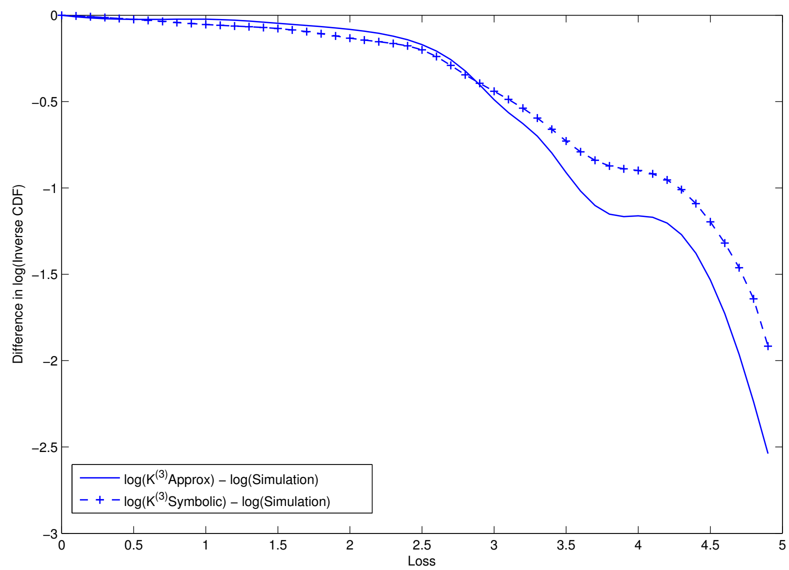

Figure 7: Difference in log(Inverse CDF) between the order 3 Symbolic approximation and the approximate analytical approximation for a Type-II portfolio at 6 month maturity

approximate the loss distribution is a very significant requirement in the design and risk management of credit linked portfolios. Assuming very generic intensity dynamics – which include the widely used doubly-stochastic and self-exciting models – we provide a small time expansion to the Laplace transform of the total portfolio loss. We then invert the transform using the Saddlepoint method. Our formulation addresses several important features of a credit portfolio, including state dependent jumps at and between defaults and stochastic volatility. We propose higher order approximations to handle the case of highly correlated portfolios over a long time horizon. We also study the applicability of other widely methods for approximating the portfolio loss, such as the delta-gamma approximation and stochastic Taylor expansions.

Appendix A

Proof of Strong Markov Property

Suppose that the space-time process (t, Z_t^X) is a strong Markov process. Consider the process $Z_t = (\Lambda_t, N_t, L_t)$ where

$$L_t = \sum_{j=1}^{n} l_j N_t^j$$

Since $h(x) = e^{-x}$ had a bounded derivative on the nonnegative interval, it is an autonomous Lipschitz continuous function on $[0, \infty)$. We now apply Theorem $V.35$ of Protter [2004] and have that (t, Z_t) is a strong Markov process. Considering the portfolio loss

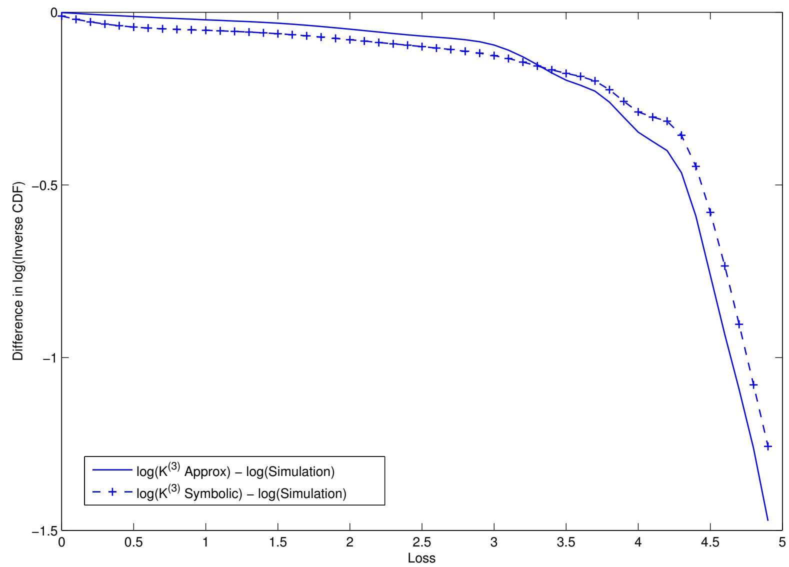

Figure 8: Difference in log(Inverse CDF) between the order 3 Symbolic approximation and the approximate analytical approximation for a Type-II portfolio at 1 Year maturity

process, L_t we have

$$\begin{aligned} |L_t| &= \left| \sum_{j=1}^{n} l_j N_t^j \right| \\ &\leq \sum_{j=1}^{n} |l_j| \\ &= L_{max} \end{aligned}$$

and therefore $|L_t|$ is a bounded process. We also have that the function $h(t, z) = e^{uz_L}$ and all its subsequent derivatives are bounded and continuous on the compact support $L_t \in [-L_{max}, L_{max}]$, and thus $h(t, z)$ is in the domain $D_{\mathcal{A}}$ of the infinitesimal generator $\mathcal{A}$. The form of the generator is given as in Section 4.3 as

$$\begin{aligned} \mathcal{A}h(t, z) = h_t(t, z) + h_z(t, z)\mu(t, \lambda, n) + \frac{1}{2} tr[h_{zz}(t, z)\sigma(t, \lambda, n)\sigma(t, \lambda, n)^T] \\ + \sum_m F_m(\lambda) \int_{\mathbb{R}^K} [h(t, z + z_m^{(1)}) - h(t, z)] d\nu_m(\epsilon) \\ + \sum_j \lambda_j(z)(1 - n_j)[h(t, z + z_j^{(2)}) - h(t, z)] \end{aligned} \tag{74}$$

where

$$z_m^{(1)} = \begin{cases} 0 & \text{for all } l \text{ components} \\ \epsilon_j & \text{for all } \lambda \end{cases}$$

and

$$z_j^{(2)} = \begin{cases} 0 & \text{for all } \lambda_j, \text{ and } n_l \text{ where } l \neq j \\ \xi_{j,l}(t) & \text{for } \lambda_l \text{ where } l \neq j \\ 1 & \text{for } n_j \\ \sum_k g^{j,k}(t,\lambda) & \text{for } l \end{cases}$$

Appendix B

Proof of Proposition 4.3

The first order expansion of the Laplace transform is given by

$$\psi^{(1)}(\Delta, u|z) = e^{uz_l}\left(1 + \sum_j \lambda_j(1-n_j)\left(e^{\sum_k g^{j,k}} - 1\right)\right)$$

The Δ-order cumulant is obtained by taking the logarithm of the above expression

$$\begin{aligned} K^{(1)}(\Delta, u|z) &= uz_L + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta \\ &= uz_L + \sum_j \lambda_j(1-n_j)\left(e^{\sum_k g^{j,k}} - 1\right)\Delta \end{aligned}$$

The second order expansion in Δ of ψ is

$$K^{(2)}(\Delta, u|z) = exp(K^{(1)}(\Delta, u|z)) \cdot \left(1 + \frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\bigg|_{\Delta=0} \Delta^2\right)$$

and the second order cumulant $K^{(2)}(\Delta, u|z)$ is found upto the second power of Δ around $\Delta = 0$ using the Taylor series of

$$\log\left(h(0,z) + \mathcal{A}h(0,z)\Delta + \frac{1}{2}\mathcal{A}^2 h(0,z)\Delta^2\right)$$

from which we get

$$K^{(2)}(\Delta, u|z) = uz_L + c_1\Delta + \left(\frac{c_1^2}{2} - c_2\right)\Delta^2$$

where $c_1 = \frac{\mathcal{A}e^{uz_L}}{e^{uz_L}}$ and $c_2 = \frac{\mathcal{A}^2 e^{uz_L}}{2e^{uz_L}}$.

Appendix C

Expressions for c_1 and c_2

$$c_1 = \mu^l u + \frac{1}{2}[\sigma\sigma^T]^{ll} u^2 + \sum_j \lambda_j(1-n_j)\left(e^{u\sum_k g^{j,k}} - 1\right)$$

which simplifies to

$$c_1 = \sum_j \lambda_j(1-n_j)\left(e^{u\sum_k l_{jk}}-1\right) \tag{75}$$

and

$$\begin{aligned}
2c_2 = {} & \mu_t^l u + \frac{1}{2}[\sigma\sigma^T]_t^{ll}u^2 + \sum_j \lambda_j(1-n_j)u\sum_k g_t^{j,k}e^{u\sum_m g^{j,m}} \\
& + [c_1 e^{uz_l}]_z\mu + \frac{1}{2}\mathrm{tr}[(c_1 e^{uz_l})_{zz}\sigma\sigma^T]e^{-uz_l} \\
& + \sum_m \iota_m\left(\sum_j \epsilon(1-n_j)(e^{u\sum_k g^{j,k}}-1)\right) \\
& + \sum_j \lambda_j(1-n_j)\left(\left(\mu_{w_{j+}}^l u + \frac{1}{2}[\sigma\sigma^T]^{ll}u^2 + \right.\right. \\
& \left.\left.\sum_l(\lambda_l+\xi_{l,j})(1-n_l)\left(e^{u\sum_k g^{l,k}}-1\right)\right)e^{u\sum_k g^{j,k}} - c1\right)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
2c_2 = {} & \sum_j \lambda_j(1-n_j)u\sum_k g_t^{j,k}e^{u\sum_m g^{j,m}} \\
& + \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k g^{j,k}}-1\right)\right]_z \mu \\
& + \frac{1}{2}tr\left[\left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k g^{j,k}}-1\right)\right]_{zz}\sigma\sigma^T\right] \\
& + \sum_m \iota_m\left(\sum_j \epsilon(1-n_j)(e^{u\sum_k g^{j,k}}-1)\right) \\
& + \sum_j \lambda_j(1-n_j)\left[\left(\sum_l(\lambda_l+\xi_{l,j})(1-n_l)\left(e^{u\sum_k g^{l,k}}-1\right)\right)e^{u\sum_k g^{j,k}} - c_1\right]
\end{aligned} \tag{76}$$

Appendix D contains additional expressions for the various partial derivatives involved in equations (75) and (76).

Appendix D

Saddlepoint Approximation

We have the following derivatives for the CGF

$$K' = v + c_1'\Delta + (-c_1c_1' + c_2')\Delta^2$$

$$K'' = c_1''\Delta + (-(c_1')^2 - c_1 c_1'' + c_2'')\Delta^2$$

$$K^{(3)} = c_1^{(3)}\Delta + \left(-3c_1'c_1'' - c_1 c_1^{(3)} + c_2^{(3)}\right)\Delta^2$$

$$K^{(4)} = c_1^{(4)}\Delta + \left(-3(c_1'')^2 - 4c_1'c_1^{(3)} - c_1 c_1^{(4)} + c_2^{(4)}\right)\Delta^2 + O(\Delta^3)$$

Derivatives w.r.t. u

$$\frac{\partial c_1}{\partial u} = J_u$$

$$\frac{\partial^2 c_1}{\partial u^2} = J_{uu}$$

$$\frac{\partial^3 c_1}{\partial u^3} = J_{uuu}$$

$$\frac{\partial^4 c_1}{\partial u^4} = J_{uuuu}$$

$$2\frac{\partial c_2}{\partial u} = J_{tu} + J_{zu}\mu + \frac{1}{2}\mathrm{tr}\left[J_{zzu}\sigma\sigma^\top\right] + A_{w_j+u} - J_u$$

$$2\frac{\partial^2 c_2}{\partial u^2} = J_{tuu} + J_{zuu}\mu + \frac{1}{2}\mathrm{tr}\left[J_{zzuu}\sigma\sigma^\top\right] + A_{w_j+uu} - J_{uu}$$

$$2\frac{\partial^3 c_2}{\partial u^3} = J_{tuuu} + J_{zuuu}\mu + \frac{1}{2}\mathrm{tr}\left[J_{zzuuu}\sigma\sigma^\top\right] + A_{w_j+uuu} - J_{uuu}$$

$$2\frac{\partial^2 c_4}{\partial u^4} = J_{tuuuu} + J_{zuuuu}\mu + \frac{1}{2}\mathrm{tr}\left[J_{zzuuuu}\sigma\sigma^\top\right] + A_{w_j+uuuu} - J_{uuuu}$$

Jump Derivatives

First Order

$$J_u = -\sum_j \lambda_j(1-n_j)\sum_k g^{jk} e^{-u\sum_l g^{jl}}$$

$$J_{tu} = \sum_j \lambda_j(1-n_j)\left(u\sum_k g_t^{jk}\sum_k g^{jk} - \sum_k g_t^{jk}\right)e^{-u\sum_k g^{jk}}$$

$$J_{yu} = \sum_j \lambda_j(1-n_j)\sum_k g^{jk}\left(1 - u\sum_k g^{jk}\right)e^{-u\sum_k g^{jk}}$$

$$J_{yyu} = \sum_j \lambda_j(1-n_j)\left\{-2u\left(\sum_k g^{jk}\right)^2 + \left(\sum_k g^{jk}\right)(1-I_{yy})\right\}e^{-u\sum_k g^{jk}}$$

where

$$I_{yy} = -u^2\left(\sum_k g^{jk}\right)^2 + u\left(\sum_k g^{jk}\right)$$

Let

$$A_{w_j+} = \sum_j \lambda_j (1-n_j) \left(J - \lambda_j \left(e^{-u \sum_k g^{jk}} - 1 \right) \right) e^{-u \sum_k g^{jk}}$$

then

$$A_{w_j+u} = \sum_j \lambda_j (1-n_j) \left(J_u + \lambda_j \sum_k g^{kl} e^{-u \sum_l g^{jl}} - I_{w_j+} \sum_k g^{jk} \right) e^{-u \sum_k g^{jk}}$$

where

$$I_{w_j+} = J - \lambda_j \left(e^{-u \sum_k g^{jk}} - 1 \right)$$

Second Order

$$J_{uu} = \sum_j \lambda_j (1-n_j) \left(\sum_k g^{jk} \right)^2 e^{-u \sum_k g^{jk}}$$

$$J_{tuu} = \sum_j \lambda_j (1-n_j) \left(2 \sum_k g_t^{jk} \sum_k g^{jk} - u \sum_k g_t^{jk} \left(\sum_k g^{jk} \right)^2 \right) e^{-u \sum_k g^{jk}}$$

$$J_{yuu} = \sum_j \lambda_j (1-n_j) \left(u \left(\sum_k g^{jk} \right)^3 - 2 \left(\sum_k g^{jk} \right)^2 \right) e^{-u \sum_k g^{jk}}$$

$$J_{yyuu} = \sum_j \lambda_j (1-n_j) \left\{ -3 \left(\sum_k g^{jk} \right)^2 + 2u \left(\sum_k g^{jk} \right)^3 - I_{yyu} \left(\sum_k g^{jk} \right) \right\} e^{-u \sum_k g^{jk}}$$

where

$$I_{yyu} = -2u \left(\sum_k g^{jk} \right)^2 + \left(\sum_k g^{jk} \right) - I_{yy} \left(\sum_k g^{jk} \right)$$

Let

$$A_{w_j+uu} = \sum_j \lambda_j (1-n_j) \left(J_{uu} - J_u \sum_k g^{jk} - 2\lambda_j \left(\sum_k g^{jk} \right)^2 e^{-u \sum_k g^{jk}} \right.$$
$$\left. - I_{w_j+u} \sum_k g^{jk} \right) e^{-u \sum_k g^{jk}}$$

where

$$I_{w_j+u} = J_u + \lambda_j \sum_k g^{jk} e^{-u \sum_k g^{jk}} - I_{w_j+} \sum_k g^{jk}$$

Third Order

$$J_{uuu} = -\sum_j \lambda_j(1-n_j)\left(\sum_k g^{jk}\right)^3 e^{-u\sum_k g^{jk}}$$

$$J_{tuuu} = \sum_j \lambda_j(1-n_j)\left(u\sum_k g_t^{jk}\left(\sum_k g^{jk}\right)^3 - 3\sum_k g_t^{jk}\left(\sum_k g^{jk}\right)^2\right) e^{-u\sum_k g^{jk}}$$

$$J_{yuuu} = \sum_j \lambda_j(1-n_j)\left(3\left(\sum_k g^{jk}\right)^3 - u\left(\sum_k g^{jk}\right)^4\right) e^{-u\sum_k g^{jk}}$$

$$J_{yyuuu} = \sum_j \lambda_j(1-n_j)\left\{5\left(\sum_k g^{jk}\right)^3 - 2u\left(\sum_k g^{jk}\right)^4 - I_{yyuu}\left(\sum_k g^{jk}\right)\right\} e^{-u\sum_k g^{jk}}$$

where

$$I_{yyuu} = -3\left(\sum_k g^{jk}\right)^2 + 2u\left(\sum_k g^{jk}\right)^3 - I_{yyu}\left(\sum_k g^{jk}\right)$$

Let

$$A_{w_j+uuu} = \sum_j \lambda_j(1-n_j)\Bigg(J_{uuu} + (J_u - 2J_{uu})\sum_k g^{jk} + 4\lambda_j\left(\sum_k g^{jk}\right)^3 e^{-u\sum_k g^{jk}}$$
$$-I_{w_j+uu}\sum_k g^{jk}\Bigg) e^{-u\sum_k g^{jk}}$$

where

$$I_{w_j+uu} = J_{uu} - J_u\sum_k g^{jk} - 2\lambda_j\left(\sum_k g^{jk}\right)^2 e^{-u\sum_k g^{jk}} - I_{w_j+u}\sum_k g^{jk}$$

Fourth Order

$$J_{uuuu} = \sum_j \lambda_j(1-n_j)\left(\sum_k g^{jk}\right)^4 e^{-u\sum_k g^{jk}}$$

$$J_{tuuuu} = \sum_j \lambda_j(1-n_j)\left(4\sum_k g_t^{jk}\left(\sum_k g^{jk}\right)^3 - u\sum_k g_t^{jk}\left(\sum_k g^{jk}\right)^4\right) e^{-u\sum_k g^{jk}}$$

$$J_{yuuuu} = \sum_j \lambda_j(1-n_j)\left(u\left(\sum_k g^{jk}\right)^5 - 4\left(\sum_k g^{jk}\right)^4\right) e^{-u\sum_k g^{jk}}$$

$$J_{yyuuuu} = \sum_j \lambda_j(1-n_j)\left\{-7\left(\sum_k g^{jk}\right)^4 + 2u\left(\sum_k g^{jk}\right)^5 - I_{yyuuu}\left(\sum_k g^{jk}\right)\right\} e^{-u\sum_k g^{jk}}$$

where

$$I_{yyuuu} = 5\left(\sum_k g^{jk}\right)^3 - 2u\left(\sum_k g^{jk}\right)^4 - I_{yyuu}\left(\sum_k g^{jk}\right)$$

Let

$$\begin{aligned} A_{w_j+uuuu} = \sum_j \lambda_j(1-n_j)\Bigg(J_{uuuu} - 3J_{uuu}\sum_k g^{jk} + 3J_{uu}\left(\sum_k g^{jk}\right)^2 \\ -J_u\left(\sum_k g^{jk}\right)^3 - 8\lambda_j\left(\sum_k g^{jk}\right)^4 \\ -I_{w_j+uuu}\sum_k g^{jk}\Bigg) e^{-u\sum_k g^{jk}} \end{aligned}$$

where

$$\begin{aligned} I_{w_j+uuu} = J_{uuu} - 2J_{uu}\sum_k g^{jk} + J_u\left(\sum_k g^{jk}\right)^2 \\ +4\lambda_j\left(\sum_k g^{jk}\right)^3 e^{-u\sum_k g^{jk}} - I_{w_j+uu}\sum_k g^{jk} \end{aligned}$$

Appendix E

Proof of Proposition 6.1

As explained in Section 6.1, we split the derivation of the error bound into two steps by first considering the polynomial part of the higher order derivatives of $K(\Delta, u|x)$ [see Equation (42)], and then handle the terms involving the jump process. Denoting the coefficients of the polynomial part of $\frac{\partial^k}{\partial \Delta^k} K(\Delta, u|x)\Big|_{\Delta=0}$ by $\alpha_{k,j}, j \in \{1, 2, \ldots, k+1\}$ and the resulting CGF as K^{poly} we have

$$\frac{1}{k!}\frac{\partial^k}{\partial \Delta^k} K^{poly}(\Delta, u|x)\Bigg|_{\Delta=0} = \sum_{j=1}^{k+1} \alpha_{k,j} u^j \qquad (77)$$

Note that $\alpha_{k,j}$ depends only on x, through $\mu(x), \sigma(x)$ and their derivatives. In order to evaluate the polynomial part of the density integral in Equation [41], which we denote as $q^{(n_2)}_{poly}(\Delta, y|x)$, we proceed as in the proof of Theorem 3 in Ait-Sahalia and Yu [2006] in which the authors provide a proof for the second order expansion. We extend the proof by considering an n_2-th order expansion. The density $q^{(n_2)}_{poly}(\Delta, y|x)$ can be written as

$$\begin{aligned}
q^{(n_2)}_{poly}(\Delta, y|x) &= \frac{1}{2\pi}\mathbf{Re}\left[\int_u e^{-\mathbf{i}uy}e^{\mathbf{i}ux+(\mathbf{i}\alpha_{2,1}u-\alpha_{2,2}u^2)\Delta}\right. \\
&\quad \left.\left(\frac{1}{2}\frac{\partial^{n_2}}{\partial\Delta^{n_2}}K(\Delta, \mathbf{i}u|x)\Big|_{\Delta=0}\Delta^2+\cdots+o(\Delta^{n_2})\right)du\right] \\
&= \frac{1}{2\pi}\mathbf{Re}\left[\int_u e^{-\mathbf{i}uy}e^{\mathbf{i}ux+(\mathbf{i}\alpha_{2,1}u-\alpha_{2,2}u^2)\Delta}\left(\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k\alpha_{k,j}(\mathbf{i}u)^j\right)du\right] \\
&= \frac{1}{2\pi}\mathbf{Re}\left[\int_u e^{-\mathbf{i}uz\Delta^{1/2}+(\mathbf{i}\alpha_{2,1}u-\alpha_{2,2}u^2)\Delta}\left(\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k\alpha_{k,j}(\mathbf{i}u)^j\right)du\right] \\
&= \frac{1}{2\pi}\mathbf{Re}\left[\int_u \left(\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))-\mathbf{i}\sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))\right)\right. \\
&\quad \left.\times e^{-\alpha_{2,2}u^2\Delta}\left(\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k\alpha_{k,j}(\mathbf{i}u)^j\right)du\right] \\
&= \frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1,\text{ j even}}^{k+1}\Delta^k\alpha_{k,j}\int_u\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}u^j du\right\} \\
&\quad +\frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1,\text{ j odd}}^{k+1}\Delta^k\alpha_{k,j}\int_u\sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}u^j du\right\}
\end{aligned} \tag{78}$$

Each of the integrals above have closed form expressions. For e.g. we have,

$$\int_{-\infty}^{+\infty}\sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}udu = e^{-\frac{z^2}{4\alpha_{2,2}}}\sqrt{\frac{\pi}{2\alpha^3_{2,2}}}z\Delta^{-1}+O(\Delta^{-1/2})$$

$$\int_{-\infty}^{+\infty}\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}du = e^{-\frac{z^2}{4\alpha_{2,2}}}\sqrt{\frac{\pi}{\alpha_{2,2}}}\Delta^{-1/2}+O(\Delta)$$

The order of these integrals is given by

$$\int_{-\infty}^{+\infty}\sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}u^j du = \begin{cases}0 & \text{if } j \text{ even}\\ O(\Delta^{-(j+1)/2}) & \text{if } j \text{ odd}\end{cases}$$

$$\int_{-\infty}^{+\infty}\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}u^j du = \begin{cases}O(\Delta^{-(j+1)/2}) & \text{if } j \text{ even}\\ 0 & \text{if } j \text{ odd}\end{cases} \tag{79}$$

Substituting the above integrals in Equation (78) we have

$$q^{(n_2)}_{poly}(\Delta, y|x) = \frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1,\text{ j even}}^{k+1}\Delta^k\alpha_{k,j}\int_u\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}u^j du\right\}$$

$$+\frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1,\text{ j odd}}^{k+1}\Delta^k\alpha_{k,j}\int_u \sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}u^j du\right\}$$

$$=\frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k\alpha_{k,j}O(\Delta^{-(j+1)/2})\right\} \quad (80)$$

We now proceed to consider terms in the higher order derivatives of $K(\Delta,u|x)$ arising from the jump component. In evaluating the non-polynomial component of the derivative expansions, we encounter additional integrals of the form

$$\frac{1}{2\pi}\mathbf{Re}\left[\int_u e^{-\mathbf{i}uy}e^{\mathbf{i}ux+(\mathbf{i}\alpha_{2,1}u-\alpha_{2,2}u^2)\Delta}f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)\right]$$

$$=\frac{1}{2\pi}\mathbf{Re}\left[\int_u e^{-\mathbf{i}uz\Delta^{1/2}+(\mathbf{i}\alpha_{2,1}u-\alpha_{2,2}u^2)\Delta}f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)du\right]$$

$$=\frac{1}{2\pi}\mathbf{Re}\left[\int_u \left(\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))-\mathbf{i}\sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))\right)\right.$$

$$\left.\times e^{-\alpha_{2,2}u^2\Delta}f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)du\right]$$

$$=\frac{1}{2\pi}\left\{\int_u \cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}\mathbf{Re}[f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)]du\right\} \quad (81)$$

$$+\frac{1}{2\pi}\left\{\int_u \sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}\mathbf{Im}[f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)]du\right\}$$

where $f(\lambda^*,\theta(u),u)$ denotes a function of λ^* and $\theta(u)$ and u. Given the structure of Λ^*, $\theta(u)$ – parameters which are determined from the explicit form of the jump component of the SDE under consideration – we can evaluate the function $f(\lambda^*,\theta(u),u)$ and hence the following integrals which are used in the computation of the error bound

$$\int_{-\infty}^{+\infty}\sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))\mathbf{Im}[f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)]e^{-\alpha_{2,2}u^2\Delta}du \quad (82)$$

and

$$\int_{-\infty}^{+\infty}\cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))\mathbf{Re}[f(\lambda^*,\theta(\mathbf{i}u),\mathbf{i}u)]e^{-\alpha_{2,2}u^2\Delta}du \quad (83)$$

The procedure to get the final error bound is as follows. For a given specification of the default intensities, we can obtain an order estimate (in Δ) for the integrals in Equations (82) and (83). Using a result proved in Theorem 3 of Ait-Sahalia and Yu [2006] we have for the real density

$$p(\Delta,y|x)=\sum_{k=0}^{n_2}g_k^{(n_2)}\Delta^{k/2}(1+\Delta^{k/2}) \quad (84)$$

where $g_k^{(n_2)}$ is a function of the parameters of the SDE. The approximate density $q^{(n_2)}(\Delta,y|x)$, consisting of the polynomial and the non-polynomial (in u) terms derived in Equations

(80) and (81) can be written as

$$q^{(n_2)}(\Delta, y|x) = \frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k \alpha_{k,j} O(\Delta^{-(j+1)/2})\right\} \tag{85}$$

$$+\frac{1}{2\pi}\left\{\int_u \cos(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}\mathbf{Re}[f(\lambda^*, \theta(\mathbf{i}u), \mathbf{i}u)]du\right\}$$

$$+\frac{1}{2\pi}\left\{\int_u \sin(u(z\Delta^{1/2}-\alpha_{2,1}\Delta))e^{-\alpha_{2,2}u^2\Delta}\mathbf{Im}[f(\lambda^*, \theta(\mathbf{i}u), \mathbf{i}u)]du\right\}$$

Comparing Equations (84) and (85), we can estimate the error bound for the CGF approximation step. Note that in the absence of jumps, Equation (85) reduces to

$$q^{(n_2)}(\Delta, y|x) = \frac{1}{2\pi}\left\{\sum_{k=2}^{n_2}\sum_{j=1}^{k+1}\Delta^k \alpha_{k,j} O(\Delta^{-(j+1)/2})\right\} \tag{86}$$

and comparing Equations (84) and (86) we have

$$q^{(n_2)}(\Delta, y|x) = p(\Delta, y|x)(1 + O(\Delta^{n_2/2}))$$

We thus have a clear characterization of the CGF error in terms of the order used in the small-time approximation, and involving the parameters of the SDE.

Appendix F

Proof of Proposition 6.2

For the sake of completeness, we consider the expansion for the density $q^{(2)}(\Delta, y|x)$ corresponding to the order-2 Laplace transform approximation and characterize the terms of the error. The method presented can easily be extended to higher order by considering the appropriate CGF expansion. We can get higher order expansions for the CGF using the infinitesimal generator as explained in Section 4. We have

$$q^{(2)}(\Delta, x + z\Delta^{1/2}|X_0 = x) = Re\left[\frac{1}{2\pi}\int_{\hat{u}-\mathbf{i}\infty}^{\hat{u}+\mathbf{i}\infty} e^{K^{(2)}(u)-u(x+z\Delta^{1/2})}du\right]$$

$$= Re\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{K^{(2)}(\hat{u}+\mathbf{i}w)-(\hat{u}+\mathbf{i}w)(x+z\Delta^{1/2})}dw\right]$$

and in order to simplify the exponent inside the integral, we consider

$$[K^{(2)}(\hat{u}+\mathbf{i}w) - (\hat{u}+\mathbf{i}w)(x+z\Delta^{1/2})] - \left[K^{(2)}(\hat{u}) - \hat{u}(x+z\Delta^{1/2})\right]$$

$$= -\mathbf{i}w(z\Delta^{1/2} - \mu(x)\Delta - \sigma^2(x)\hat{u}\Delta) - \frac{1}{2}\sigma^2(x)\Delta w^2$$

$$+\ln\left(\frac{1 + (\alpha_{2,3}(\hat{u}+\mathbf{i}w)^3 + \alpha_{2,2}(\hat{u}+\mathbf{i}w)^2 + \alpha_{2,1}(\hat{u}+\mathbf{i}w))\Delta^2}{1 + (\alpha_{2,3}\hat{u}^3 + \alpha_{2,2}\hat{u}^2 + \alpha_{2,1}\hat{u})\Delta^2}\right)$$

$$+\ln\left(\frac{g(\lambda^*,\theta(\hat{u}+\mathbf{i}w),\hat{u}+\mathbf{i}w)}{g(\lambda^*,\theta(\hat{u}),\hat{u})}\right)$$

Using the above result, we thus have for the density

$$q^{(2)}(\Delta,x+z\Delta^{1/2}|X_0=x)=e^{K^{(2)}(\hat{u})-\hat{u}(x+z\Delta^{1/2})}\times$$

$$Re\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\mathbf{i}w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta)-\frac{1}{2}\sigma^2(x)\Delta w^2\right)\times\right.$$
$$\left.\left(\frac{1+(\alpha_{2,3}(\hat{u}+\mathbf{i}w)^3+\alpha_{2,2}(\hat{u}+\mathbf{i}w)^2+\alpha_{2,1}(\hat{u}+\mathbf{i}w))\Delta^2}{1+(\alpha_{2,3}\hat{u}^3+\alpha_{2,2}\hat{u}^2+\alpha_{2,1}\hat{u})\Delta^2}\right)\left(\frac{g(\lambda^*,\theta(\hat{u}+\mathbf{i}w),\hat{u}+\mathbf{i}w)}{g(\lambda^*,\theta(\hat{u}),\hat{u})}\right)dw\right]$$

in which each of the terms simplifies in the case k is even to

$$Re\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\mathbf{i}w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta)-\frac{1}{2}\sigma^2(x)\Delta w^2\right)(\mathbf{i}w)^k dw\right]$$
$$=\frac{(-1)^{k/2}}{2\pi}\int_{-\infty}^{\infty}\cos\left[w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta)\right]\exp\left[-\frac{1}{2}\sigma^2(x)\Delta w^2\right]w^k dw$$

and in the case k is odd, simplifies to

$$Re\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\mathbf{i}w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta)-\frac{1}{2}\sigma^2(x)\Delta w^2\right)(\mathbf{i}w)^k dw\right]$$
$$=\frac{(-1)^{(k-1)/2}}{2\pi}\int_{-\infty}^{\infty}\sin\left[w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta)\right]\exp\left[-\frac{1}{2}\sigma^2(x)\Delta w^2\right]w^k dw$$

As in Equation (79) we have

$$\int_{-\infty}^{+\infty}\sin(w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta))e^{-\alpha_{2,2}w^2\Delta}w^j dw=\begin{cases}0 & \text{if } j \text{ even}\\ O(\Delta^{-(j+1)/2}) & \text{if } j \text{ odd}\end{cases}$$
$$\int_{-\infty}^{+\infty}\cos(w(z\Delta^{1/2}-\mu(x)\Delta-\sigma^2(x)\hat{u}\Delta))e^{-\alpha_{2,2}w^2\Delta}w^j dw=\begin{cases}O(\Delta^{-(j+1)/2}) & \text{if } j \text{ even}\\ 0 & \text{if } j \text{ odd}\end{cases}$$

The saddlepoint $\hat{u}$ can be got by solving

$$x+z\Delta^{1/2}=\frac{\partial}{\partial u}K^{(2)}(\Delta,\hat{u}|x) \tag{87}$$

From Equation (47), we take the expansion of $K^{(2)}$ in terms of μ and σ. Expanding all terms in Δ to order 2, and gathering terms we have

$$K^{(2)}(\Delta,u|x)=u\left(x+\Delta\mu(x)+\frac{\Delta^2\mu(x)\mu'(x)}{2}+\frac{\Delta\sigma^2(x)\mu''(x)}{4}\right)$$
$$+u^2\left(\frac{\Delta\sigma^2(x)}{2}+\frac{\Delta^2\sigma^2(x)\mu'(x)}{2}+\frac{\Delta^2(\sigma^2)'(x)}{4}+\frac{\Delta^2\sigma^2(x)(\sigma^2)'(x)}{8}\right)$$

$$+\frac{u^3\Delta^2\sigma^2(x)(\sigma^2)'(x)}{4}$$

Equation (87) is thus a quadratic in u. Solving the equation and choosing the root with the leading term matching the first order saddlepoint ($\frac{z}{\sigma^2(x)\Delta^{1/2}}$), we have $u = \hat{u}^{(2)} + O(\Delta^{1/2})$ where

$$\hat{u}^{(2)} = \frac{z}{\sigma^2(x)\Delta^{1/2}} - \frac{2\mu(x)\sigma(x) + 3\sigma'(x)z^2}{2\sigma^3(x)}$$

Thus, we have

$$z\Delta^{1/2} - \mu(x)\Delta - \sigma^2(x)\hat{u}\Delta = \frac{3\sigma'(x)z^2\Delta}{2\sigma(x)}\left(1 + O(\Delta^{1/2})\right)$$

With these results, we can calculate the error order for $q^{(0,2)}$ as

$$q^{(0,2)}(\Delta, x + z\Delta^{1/2}|x) = \frac{1}{\sqrt{2\pi\sigma^2(x)\Delta}} e^{K^{(2)}(\hat{u}) - \hat{u}(x+z\Delta^{1/2})}\left[1 - \frac{3\sigma'^2(x)z}{2\sigma(x)}\sqrt{\Delta} + O(\Delta)\right] \times$$
$$\mathbf{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\mathbf{i}w\frac{3\sigma'(x)z^2\Delta}{2\sigma(x)} - \frac{1}{2}\sigma^2(x)\Delta w^2\right)\frac{g(\lambda^*, \theta(\hat{u} + \mathbf{i}w), \hat{u} + \mathbf{i}w)}{g(\lambda^*, \theta(\hat{u}), \hat{u})}dw\right]$$

From the generalized saddlepoint formula (Equation 46), we have

$$p^{(0,2)}(\Delta, x + z\Delta^{1/2}|x) = \frac{1}{\sqrt{2\pi\sigma^2(x)\Delta}} e^{K^{(2)}(\hat{u}) - \hat{u}(x+z\Delta^{1/2})}\left[1 - \frac{3\sigma'^2(x)z}{2\sigma(x)}\sqrt{\Delta} + O(\Delta)\right]$$

Thus, we conclude that

$$p^{(0,2)}(\Delta, x + z\Delta^{1/2}|x) = q^{(0,2)}(\Delta, x + z\Delta^{1/2}|x)(1 + O(\Delta)) \times$$
$$\mathbf{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\mathbf{i}w(z\Delta^{1/2} - \mu(x)\Delta - \sigma^2(x)\hat{u}\Delta) - \frac{1}{2}\sigma^2(x)\Delta w^2\right) \times \right.$$
$$\left.\frac{g(\lambda^*, \theta(\hat{u} + \mathbf{i}w), \hat{u} + \mathbf{i}w)}{g(\lambda^*, \theta(\hat{u}), \hat{u})}dw\right]$$

We can get higher order proofs, by starting with the appropriate order CGF expansion.

Appendix G

Error characterization of the CGF Step - Poisson Case

In this Appendix, we consider the setting as in Section 3 and compute the error bound of the CGF approximation step, following the procedure explained in Appendix E. Since the $\mu(x)$ and $\sigma(x)$ components of the SDE in Equation (39) are zero in this case, we have

$$\frac{\partial^k}{\partial\Delta^k}K^{poly}(\Delta, u|z) = 0, \forall k \tag{88}$$

Thus only the jump components remain in Equation (85). Given the intensity λ is constant, the moment generating function of the jump component is

$$\theta(u) = e^u \tag{89}$$

We consider the 3-rd order expansion to the CGF and from the results of Appendix D, we have

$$\begin{aligned}
\left.\frac{\partial K(\Delta, u|z)}{\partial \Delta}\right|_{\Delta=0} &= n\lambda(e^u - 1) \\
\left.\frac{\partial^2 K(\Delta, u|z)}{\partial \Delta^2}\right|_{\Delta=0} &= -n\lambda^2 e^u(e^u - 1) \\
\left.\frac{\partial^3 K(\Delta, u|z)}{\partial \Delta^3}\right|_{\Delta=0} &= n\lambda^3 e^{2u}(e^u - 1)
\end{aligned}$$

From Equation (81), we have the following for the approximate density

$$\begin{aligned}
q^{(3)}(\Delta, y|z) = \frac{1}{2\pi}\mathbf{Re}\Bigg[\int_u & e^{-\mathbf{i}uy} e^{\mathbf{i}ux + (\mathbf{i}\alpha_{2,1}u - \alpha_{2,2}u^2)\Delta} \\
& \times \left(-\frac{1}{2}n\lambda^2 e^{\mathbf{i}u}(e^{\mathbf{i}u} - 1)\Delta^2 + \frac{1}{6}n\lambda^3 e^{2\mathbf{i}u}(e^{\mathbf{i}u} - 1)\Delta^3\right) du\Bigg]
\end{aligned}$$

which simplifies to

$$\begin{aligned}
q^{(3)}(\Delta, y|z) = -\frac{1}{2\pi}\Bigg\{\int_u \cos(u(z\Delta^{1/2}))e^{-u^2\Delta}\Bigg[&\frac{1}{2}n\lambda^2(\cos(2u) - \cos(u))\Delta^2 \\
&+ \frac{1}{6}n\lambda^3(\cos(3u) - \cos(2u))\Delta^3\Bigg] du\Bigg\} \\
+ \frac{1}{2\pi}\Bigg\{\int_u \sin(u(z\Delta^{1/2}))e^{-u^2\Delta}\Bigg[&\frac{1}{2}n\lambda^2(\sin(2u) - \sin(u))\Delta^2 \\
&+ \frac{1}{6}n\lambda^3(\sin(3u) - \sin(2u))\Delta^3\Bigg] du\Bigg\}
\end{aligned} \tag{90}$$

The order of the integrals is given by

$$\begin{aligned}
\int_u \sin(u(z\Delta^{1/2}))e^{-u^2\Delta} sin(ku) &= O(\Delta^{-1}) \\
\int_u \cos(u(z\Delta^{1/2}))e^{-u^2\Delta} cos(ku) &= O(\Delta^{-1})
\end{aligned}$$

and thus we can have the error order for Equation (90) as $O(\Delta^2)$ which agrees with the result in Equation (13).

Appendix H – Alternative Methods

Delta – Gamma Approximation

In this appendix, we analyze the widely used Delta Gamma approximation to calculate the distribution of portfolio value and total loss for derivative portfolios. For the case of a pure diffusion model (no-jumps), Cardenas et al. [1997] and Rouvinez [1997] derive an analytical method to compute the characteristic function of the delta-gamma approximation of the change in the portfolio value. They then invert the function using inverse Fourier transforms to obtain the portfolio value distribution (the density in the case of Cardenas et al. [1997]).

Duffie and Pan [2001] consider a risk setting of a multi-factor jump diffusion for default intensities and asset returns, under which between-jump returns are correlated Brownian motions, with return jumps at Poisson arrivals that are jointly distributed. The portfolios they consider include options and other derivatives. The risk setting allows for fat-tailed and skewed return distributions. The analytically estimate the characteristic function of the delta-gamma approximation for the change in the portfolio value and use numerical Fourier inversion to obtain the portfolio value distribution. Glasserman et al. [2000] consider a setting of heavy tailed risk factors to which they apply the delta-gamma approach.

Consider the process (t, Z^X) from Section 4. For ease of exposition, we define the simplified strong Markov process (t, U) where $U = (\Lambda, N)$ and the augmented space-time vector process (t, V) where $V = (\Lambda, N, L)$. While L is not a strong Markov process, as proved before, V is. The time T value of the total portfolio loss process is thus $V_L(T)$ and we write it as $V_L(\Lambda_T, N_T) = V_L(W_T)$. Using a delta-gamma approximation for the loss component of the markov process we have:

$$V_L(W_T) \approx V_L^{\Delta,\Gamma}(W_T) = V_L(W_0) + \Delta(W_T - W_0) + \frac{1}{2}(W_T - W_0)^\intercal \Gamma (W_T - W_0)$$

where Δ and Γ are the first (gradient) and second (Hessian) derivatives of $V_L(\cdot)$ evaluated at W_0. We can rewrite the above equation as:

$$V_L(W_T) \approx V_L^{\Delta,\Gamma}(W_T) = A + B \cdot W_T + \frac{1}{2} W_T^\intercal \Gamma W_T \tag{91}$$

where

$$A = V_L(W_0) - \Delta \cdot W_0 + \frac{1}{2} W_0^\intercal \Gamma W_0, \qquad B = \Delta - \Gamma W_0$$

The characteristic function of $V_L(W_T)$ defined by $\psi(u) = E(e^{\mathbf{i}uV_L(W_T)})$ is composed of three different components:

$$\psi(u) = E(e^{\mathbf{i}uA}) E(e^{\mathbf{i}uBW_T}) E(e^{\frac{1}{2}\mathbf{i}uW_T^\intercal \Gamma W_T}) \tag{92}$$

Conditioning on the number of jumps, Duffie and Pan [2001] make a Gaussian approximation on W_T with the mean and variances defined by the intensity dynamics. They then compute the Greeks for options and other derivatives, conditional on the jumps as in Glasserman and Zhao [1999], and use the methods suggested in Davies [1973] to compute the conditional characteristic function. For time horizons of up to a few weeks, the Gaussian approximation is reasonable. However, in our work we are interested in the behavior of the portfolio loss over a much longer time horizon. The distribution of the process W_T at longer time horizons for our generic intensity model is not available in any closed form. From Equation (92) we see that the problem of estimating the characteristic function of the portfolio loss is not simplified using this approach. We still have to compute the characteristic function of the process W_T and estimate the sensitivities of W_T (Δ and Γ). The consideration of longer time horizons and generic jump-diffusion intensity dynamics make the delta-gamma approximation unsuitable for our problem.

Stochastic Taylor Expansions

Bruti-Liberati and Platen [2005] analyze strong approximations for pure jump diffusion processes. By using a jump-adapted time discretization, they derive bias free expansions when the underlying SDE for the jump process is known. They also propose higher order discrete time strong approximations which are independent of the jump intensity. For any measurable function $f : \mathbb{R} \to \mathbb{R}$, and a given adapted counting process $N = \{N_t, t \in [0, T]\}$, the stochastic Taylor expansion for $f(N_t)$ is given by:

$$f(N_t) = \sum_{k=0}^{n} (\tilde{\Delta}_N)^k f(N_0) \binom{N_t}{k} + \bar{R}_{n+1}(t) \tag{93}$$

where

$$\bar{R}_{n+1}(t) = \int_{(0,t]} \dots \int_{(0,s_2)} (\tilde{\Delta}_N)^{n+1} f(N_{s_1-}) dN_{s_1} dN_{s_2} \dots dN_{s_{n+1}}$$

for $t \in [0, T]$ and $n \in \{0, 1, \dots\}$, where $(\tilde{\Delta}_N)^0 f(N_0) = f(N_0)$. The remainder term includes the terms which occur when there are more than n jumps in the counting process N_t. Consequently, if there is a small probability that more than n jumps occur over the given time period, then the truncated Taylor expansion can be expected to be quite accurate under any reasonable criterion. An equivalent expansion can be derived in the case of a pure jump diffusion process X_t. Bruti-Liberati and Platen [2005] show that the popular Euler scheme (refer Protter [2004]) is a special case of the stochastic Taylor expansion with $n = 1$.

Choosing N_t to be the number of defaults in the portfolio, Equation (93) can be used to approximate $E(f(N_t))$ in terms of the factorial moments ($E\binom{N_t}{k}$). The n^{th} order expansion for $E(f(N_t))$ is given by:

$$E(f(N_t)) \approx \sum_{k=0}^{n} (\tilde{\Delta}_N)^k f(N_0) E \binom{N_t}{k} \tag{94}$$

Potential applications of this method include computing the portfolio loss distribution (by choosing $f(N_t) = \mathbf{1}(N_T > x)$), without using the transform inversion method, and in pricing tranches (by choosing $f(N_t) = (N_t - l)^+$).

The computation of the factorial moments of N_t is quite involved. For the case of doubly-stochastic Poisson models of default intensity, we can proceed as in Daley and

Vere-Jones [2003] and Ball and Milne [2005]. Giesecke and Kim [2010] derive analytical expressions for the mark-to-market value of a portfolio of loans and credit derivatives assuming that the Laplace transforms of the cumulative intensities are analytically tractable. For the case of affine jump diffusion processes, Duffie et al. [2000] show that under a few technical regularity conditions, the extended transforms of the intensity processes remain exponentially affine functions, with the affine coefficients satisfying a system of Ricatti ODEs. The system of ODEs thus formed do not usually admit closed-form solutions. We usually have to resort to computing the solutions numerically using methods such as the Runge-Kutta method. The analysis can also be extended to quadratic diffusion processes. Filipovic et al. [2004] and Leippold and Wu [2002] found that the empirical tension that is created by the linearity and non-negativity assumptions of the affine framework can be alleviated by considering a quadratic framework in which the risk factors follow quadratic diffusion processes.

Choosing a risk setting of doubly-stochastic Poisson models, computation of the factorial moments can be reduced to the problem of estimating higher order moments of the counting process. As in Giesecke and Kim [2010], this involves the computation of the Laplace transforms of the process. The total loss of the portfolio does not necessarily follow an affine jump-diffusion, even if the relevant risk factors are assumed to be of this form. In order to compute the Laplace transform, we would have to use a method as in Merino and Nyfeler [2002], where we use the conditional independence property of the default intensities to compute a conditional transform, and then integrate against the common factor to get the unconditional Laplace transform. An additional issue is that in order to estimate the portfolio loss over a medium time horizon (3 months to a year) for a large portfolio of loans, we would have to fix a high value of n in Equation (94). From Giesecke and Kim [2010], we see that the expressions for the moments get intractable after $n = 3$. These issues affect the applicability of this expansion method in our problem setting.

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