Multi Factor Models for the Commodities Futures Curve:
Forecasting and Pricing

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Abstract—The term structure is a very important construct which defines the relationship between the spot and future prices of an asset for future delivery dates. In this paper, we develop a three factor term structure model for one of the most developed commodity futures market - the crude oil market. We start our analysis by reviewing well known one and two-factor models for the futures curve, and then go on to propose a three factor model to better capture the dynamics of the term structure. Several approaches for three factor models that have been proposed in literature assume a stochastic mean-reverting process for the interest rate, which is modeled as the third factor. A common shortcoming with this approach is the poor calibration performance arising from having to use both commodities and bond prices in the calibration process. We address this issue by proposing to use the long-term rate of return of the security as the third factor and compare the performance of our approach with other models. The analytical tractability of this problem is discussed and efficient numerical techniques to solve these models are considered. We run simulations to test the ability of our model to replicate real world futures price curves. We also analyze the calibration process and propose efficient ways to tackle this problem.

I. INTRODUCTION

The term structure defines the relationship between the spot and future prices of an asset for future delivery dates. The term structure, a fundamental construct in the futures market, serves two main purposes: a) to manage price risk, and b) to help in price discovery and market depth. It contains information available in the market and also synthesizes the operator’s expectations about the market. Countries, corporations, as well as other investors and speculators use the futures market for hedging exposures to the physical market, to adjust the stock level or the production rate, and for speculation purposes by undertaking arbitrage transactions on commodities derivatives. Commodities markets have been rapidly evolving and there by the term structure of commodities becomes even more important. In this paper, we develop multi-factor term structure models for one of the most developed commodity futures market - the crude oil market. The crude oil market has been increasing in volume and the maturities of the contracts traded has been steadily increasing.

Due to the well established futures market for crude oil, the prices and volume of the various contracts are publicly available. This is in contrast to most other commodities markets, which are traded over-the-counter (OTC), and where the information for far maturities is private and given by forward prices. This unique feature of the crude oil futures market makes it possible to perform empirical studies of the dynamics of the variables influencing the market.

The most important feature in the commodities term structure is the difference in behavior of the short-term and long-term contracts. Short term contracts tend to be much more volatile than long-term contracts. This phenomenon, called the Samuelson effect [3], is explained by the observation that the effect of shocks on short-term contracts is higher than the effect on the neighboring higher term contract; the effect of a shock reduces as the maturity of the contract increases.

In order to explain this unique behavior of commodities futures prices a dynamic theory of the term structure is needed which both models the impact of various factors on the term structure and considers the relationship between short and long-term contracts. Several factor models for the commodities term structure have been proposed in literature [4],[5],[7],[8],[9],[10],[11],[12]. Four different factors are generally used: the spot price (S), the convenience yield (χ), the interest rate (r), and the long-term price (η). In this paper, we propose a three factor model to better capture the dynamics of the term structure. Several approaches for three factor models that have been proposed in literature assume a stochastic mean–reverting process for the interest rate, which is modeled as the third factor. A common shortcoming with this approach is the poor calibration performance arising from having to use both commodities and bond prices in the calibration process. We address this issue by proposing to use the long-term rate of return of the security as the third factor and compare the performance of our approach with other models. The analytical tractability of this problem is discussed and efficient numerical techniques to solve these models are considered. We run simulations to test the ability of our model to replicate actual market data and analyze the impact of the factors used in the model. We also analyze the calibration process and propose efficient ways to tackle this problem.

The rest of this paper proceeds as follows. Section II is devoted to the analysis of well known factor models for the futures curve. Section III describes our model and the analytical results. In Section IV, we describe the data used to calibrate the model and the associated parameter estimation procedures. A few further applications are summarized in this section and finally Section V concludes the paper.

II. EXISTING MODELS

A. One-Factor Models

The spot price is usually considered to be the primary factor in determining futures prices, and therefore virtually
all one-factor models use spot price. Several one-factor models have been proposed in literature and are differentiated by the assumptions on the dynamics of the spot price. In [12], [8], and [9], the authors models the spot price as a geometric Brownian motion process, whereas [4], [6], [10] assume a mean reverting process for the spot price.

A widely used one-factor model was developed by Brennan and Schwartz [12], in which the spot price is assumed to follow a geometric Brownian motion:

$$dS(t) = \mu S(t) dt + \sigma_S S(t) dz$$  \hspace{1cm} (1)

where $S(t)$ is the spot price, $\mu$ the drift of the spot price, $\sigma_S$ the volatility of the spot price, and $dz$ is a Wiener process associated with $S(t)$.

Brennan and Schwartz [12] show that the solution to this equation is given by the Feynman-Kac solution:

$$F(S,t,T) = Se^{(-r-c)T}$$  \hspace{1cm} (2)

While the simplicity of this model makes it quite tractable, it does not capture the influence of producers and consumers in the commodities market. When the spot price is high, producers tend to increase their production rate and consumers tend to use their surplus stocks; thus reducing the spot price. When the spot price is lower than historical levels, consumers, anticipating an increase in prices, increase their stocks and producers decrease their production rate causing the price of the commodity to increase. Thus, several one-factor have been proposed ([4], [6], [10]) that assume that the spot price follows a mean reverting process. Schwartz [4] models the spot price as:

$$dS(t) = \kappa S(t)(\mu - \ln(S))dt + \sigma_S S(t) dz$$  \hspace{1cm} (3)

where $\mu$ is the long run mean of the spot price, and $\kappa$ is the rate of mean reversion.

Schwartz [4], shows that the futures price $F(S,T)$ satisfies the following equation:

$$\ln F(S,T) = e^{-rT} \ln S(T) + (1 - e^{-\kappa T})\mu + \frac{\sigma_S^2}{4\kappa}(1 - e^{-2\kappa T})$$

Figure 1 illustrates the behavior of the Schwartz model for different values of mean reversion ($\kappa$). We note that as $\kappa$ increases, the term structure reverts to its long run mean at a faster rate.

### B. Two-Factor Models

One-factor models prove to be too simplistic for modeling the commodities term structure. With two-factor models it is possible to develop curves that resemble actual term structures more realistically. As in the one-factor case, most of the two-factor models use spot price as one of the factors. Common choices for the second factor include the convenience yield [4], [6] and the long-term price [9], [11].

Schwartz [4] presents a term structure model based on spot prices and convenience yields, with their dynamics as follows:

$$dS(t) = (\mu - C)S(t) dt + \sigma_S S(t) dz$$

$$dC(t) = \kappa(\alpha - C(t)) dt + \sigma_C dz$$

with $E(dSdz) = \rho dt$

where $S(t)$ is the spot price of the commodity, $\mu$ the drift of the spot price, $C$ the convenience yield, $\sigma_S$, $\sigma_C$ the volatilities of the spot price and convenience yield respectively. Schwartz justifies the use of a mean reverting convenience yield by observing that convenience yields are related inversely to spot prices. If the convenience yield is high, the stocks are too rare, and operators will attempt to increase them. A similar explanation holds for a convenience yield that is low. Under these dynamics the futures price satisfies the following partial differential equation:

$$\frac{1}{2} \sigma_S^2 F_{SS} + \sigma_S \sigma_C F_{SC} + \frac{1}{2} \sigma_C^2 F_{CC} + (r-c)SF_S + (\kappa (\alpha - \delta) - \lambda)F_C - Ft = 0$$

The solution to the above partial differential equation is given by Bjerkund [14] and Jamshidian [13]. Figures 2 and 3 illustrate the behavior of the model with varying convenience yields and mean reversion rates respectively. Finally, we show the evolution of volatility of futures returns ($\sigma_r$) with the time of maturity of the futures contract in Figure 4. Schwartz [4] shows that the volatility of returns evolves as:

$$Vol_F(T) = \sigma_S^2 + \sigma_C^2 \frac{(1 - e^{-\kappa T})^2}{\kappa^2} - 2\rho \sigma_S \sigma_C \frac{(1 - e^{-\kappa T})}{\kappa}$$

An alternative to [4] is presented by Schwartz and Smith in [11]. The authors model the long-term price of the futures contract as a second state variable. They consider
the following dynamics:
\[
\log S_t = \chi_t + \xi_t \\
\text{(4)}
\]
\[
d\chi_t = -\kappa \chi_t dt + \sigma_\chi dz_\chi \\
d\xi_t = \mu \xi_t dt + \sigma_\xi dz_\xi
\]

In equation 4, \( \chi_t \) represents the short-term deviations from \( \xi_t \), the equilibrium price level. The short-term deviations follow an Ornstein-Uhlenbeck process with mean 0; the long-run equilibrium price follows a Brownian motion. The authors observe that the short-term end of the term structure is mainly affected by differences in supply and demand and other market frictions, whereas the long end of the term structure is influenced by macroeconomic variables such as inflation, GDP growth, and fiscal/monetary policy. The Brownian motions \( dz_\chi \) and \( dz_\xi \) are correlated with \( E[dz_\chi dz_\xi] = \rho dt \). Figure 5 plots the behavior of the model with varying mean reversion rate \( \kappa \) of the short-term factor. We note that a higher \( \kappa \) results in a lower price, since short-term deviations disappear more quickly. Figures 6 and 7 illustrate the effect of changes in \( \sigma_\chi \) and \( \sigma_\xi \) respectively. We also plot the long run volatilities in the figures. As the volatility of the factors increases, we note that the curves revert to their long-term values faster.

**C. Three Factor Models**

While two-factor models allow more refined term structures, they do not model the dynamics of the interest rate or the long run return of the commodity. Schwartz [4] presents a three factor model where the interest rate is chosen as a third factor and is modeled as a Vasicek process [15]. The calibration process involves separately calibrating the short rate interest model using bond data and using commodity futures to obtain parameters for the other two factors. The Kalman filter [16] approach used in [4], while analytically tractable, involves too much computational complexity to be implemented as a model which could be used to obtain new parameters with the arrival of live market data. This model also exhibits poor calibration performance for two reasons: firstly, there is an issue with choosing the appropriate securities to calibrate the interest rate model. The term structure of interest rates used for valuing other derivatives is not

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**Fig. 3.** Effect of Mean Reversion Rate (\( \kappa \)).

**Fig. 4.** Volatility of Futures Returns.

**Fig. 5.** Variation of spot price with \( \kappa \).

**Fig. 6.** Variation of spot price with \( \sigma_\chi \).

**Fig. 7.** Variation of spot price with \( \sigma_\xi \).
necessarily the best choice for commodity futures contracts, and secondly, the disconnect between the two calibration processes leads to introduction of significant bias in the estimated parameters [17].

The use of long-term rate of return of the commodity contract as a factor, has previously been suggested in literature [9], [11]. However, there are no three-factor models which incorporate this factor. In the next section, we propose a three factor model that uses spot rates, convenience yields, and the long run rate of return as its three factors and propose efficient ways to handle the parameter estimation problem.

III. OUR MODEL

We now describe our proposed three-factor model. We define as state variables, the spot price (S), the convenience yield (η), and the long run return of the futures contract (η).

The state variables have the following dynamics:

\[ dS(t) = (\mu - \chi(t))S(t)dt + \sigma_S S(t)d\zeta \]

\[ d\eta(t) = \delta(\eta(t) - \overline{\eta})dt + \sigma_\eta d\zeta \]

\[ E(d\zeta d\zeta) = \rho_{\chi\eta} dt \]

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where \( \rho_{\chi\eta} \), \( \rho_{\chi \zeta} \), and \( \rho_{\eta \zeta} \) are the correlations between the state variables (S, \( \chi \) and \( \eta \)). \( \mu \) the long-run total return of the commodity (including price appreciation and convenience yield), \( \chi \) the mean reversion coefficient, \( \alpha \) the long-run convenience yield, \( \delta \) the mean reversion coefficient of the long-run price return of the commodity, \( \overline{\eta} \) the long-run average of the long-run price return, and \( \sigma_S, \sigma_\chi, \sigma_\eta \) the volatilities of the three state variables.

From equation 6 we see that the spot price is dependent on two stochastic factors; the long-run total return of the commodity (which includes price appreciation and the instantaneous convenience yield) and the mean convenience yield \( \chi \). The long-run price return of the commodity price \( \eta(t) \) is also mean reverting at a rate \( \delta \). This factor implicitly captures the effect of interest rate and thus we do not have to have to contend with security selection or calibration issues in our model as is the case with other models.

By Ito’s Lemma, we can show that the futures price \( F(S, \chi, \eta, t) \) satisfies the following partial differential equation:

\[ \frac{1}{2} \sigma_S^2 S_F S_F + \frac{1}{2} \sigma_\chi^2 S_F S_F + \frac{1}{2} \sigma_\eta^2 S_F S_F + \psi_{\eta \eta} S_F S_F + (\mu - \delta - \lambda_S) S_F = -(\chi(\chi - \alpha) - \lambda_\chi) F + \delta(\eta - \overline{\eta})(\eta - \overline{\eta}) F \]

where \( \lambda_S, \lambda_\chi, \) and \( \lambda_\eta \) are the market prices of risk associated with \( S, \chi, \) and \( \eta \) respectively, and \( \psi_{\eta \eta}, \psi_{\eta \chi}, \psi_{\eta S} \) are the covariances between the state variables (\( S, \chi, \) and \( \eta \)). Unlike other securities, commodities spot contracts are not traded. Hence, we need to include \( \lambda_\eta \) in our analysis.

We use results from Bjerksund [14], Jamshidian [13], and Schwartz [7] to derive a closed form solution for the futures price which is given by:

\[ \log(F(S, \chi, \eta, t)) = \log(S(t) - \lambda_\delta) \]

\[ + \frac{\delta_\eta - \lambda_\eta + \psi_{\eta \eta}(\delta t - 1 + e^{-\delta t})}{\delta^2} \]

\[ + \frac{\chi_\chi - \psi_{\eta \chi}(\kappa t - 1 + e^{-\kappa t})}{\kappa^2} \]

\[ + \frac{\sigma_\eta^2}{2 \delta^2}(-e^{-2\delta t} + e^{-2\delta t} + 2\delta t - 3) \]

\[ - (\chi - \alpha)(1 - e^{-\kappa t}) + \frac{\eta - 1 - e^{-\delta t}}{\delta} \]

\[ + \frac{\sigma_\chi^2}{4 \kappa^2}(-e^{-2\kappa t} + e^{-2\kappa t} + 2\kappa t - 3) + \zeta(t) \]

where \( \zeta(t) = \frac{\psi_{\eta \eta}(\kappa \delta t + \kappa \delta e^{-\kappa t} + \delta^2 e^{-\kappa t})}{\kappa^2 \delta^2} - \kappa \delta e^{-\kappa t + \delta^2} - \kappa^2 - \kappa \delta - \delta^2 \]

\[ + \kappa^2 \delta^2 + \kappa e^{-\delta t} + \kappa \delta e^{-\delta t} \]

We again use Ito’s Lemma to derive the volatility \( (\nu_F(t)) \) of the futures returns:

\[ \nu_F(t) = \sigma^2_S + \left( \frac{\chi_\chi - 1 - e^{-\kappa t}}{\kappa^2} \right)^2 \left( \frac{\sigma_\eta^2}{\delta^2} \right)^2 \]

\[ -2 \psi_{\eta \eta} \frac{1 - e^{-\kappa t}}{\kappa} \frac{1}{\delta} + 2 \psi_{\eta \chi} \frac{1 - e^{-\delta t}}{\delta} \]

and where we also have the following expression for the long run volatility of the futures contract:

\[ \lim_{t \to \infty} (\nu_F(t)) = \sigma^2_S + \frac{\sigma^2_\chi}{\kappa^2} + \frac{\sigma^2_\eta}{\delta^2} \]

\[ + \frac{2 \psi_{\eta \chi}}{\kappa} - \frac{2 \psi_{\eta \eta}}{\delta} \]

The presence of such a limit not surprising given the mean reverting dynamics for the spot price and the convenience yield.

Our model is characterized by the parameter vector \( \mathcal{P} \):

\[ \mathcal{P} = [\mu, \sigma_S, \sigma_\chi, \sigma_\eta, \kappa, \delta, \overline{\eta}, \psi_{\eta \chi}, \psi_{\eta S}, \lambda_S, \lambda_\chi, \lambda_\eta] \]

IV. DATA & PARAMETER ESTIMATION

Having described the model, we now describe the data used for parameter estimation. The data used to calibrate and estimate our model consisted of all future contracts traded on NYMEX from 2000/01/13 to 2010/03/02 for West Texas Intermediate (WTI) grade light sweet crude. This is a widely traded futures contract with a very mature term structure. Currently, contracts extending up to 8 years are traded for WTI NYMEX. Figure 8 plots the time series for CL5, CL30 and CL60 contracts, which correspond to the futures contracts expiring in 5, 30 and 60 months respectively.

In order to test the in-sample and out-of-sample performance of our model, we split the data into two different sets; the in-sample set from 2000/01/13 to 2007/01/13 and the out-of-sample set from to 2009/01/01 to 2010/03/02.

For estimating the parameters of our model, we employ a two-step iterative procedure. In the first step, based on
initial estimates of the parameters, the algorithm estimates the values of the state variables – spot price, convenience yield and long-term return – most suited to the data for each trading date. The series of estimated state variables are used in the second step to obtain a better estimate for the parameter values. We then keep iterating until the parameter estimates converge. While such a process is not guaranteed to be efficient for a generic problem, the high degree of temporal likelihood present in the term structures between one day and the next, makes this procedure well adapted to our problem. In simulations, our model quickly converged irrespective of the starting values of the state variables. Once the algorithm is started it uses new data to obtain better estimates for the term structure. Thus, our algorithm is very well suited to be implemented in an environment where we have live market data feeds, making it an attractive option for live term structure estimation. We also note that our model uses all the trading data available without any clustering. Clustering and aggregation is a common problem with other calibration methodologies. In particular, methods like Kalman filters have issues with missing data in the time series – an issue with commodities futures as some of the market data values may be stale due to low liquidity or due to high interest in a nearby contract – and most implementations using these methods aggregate data for nearby maturities to tackle this issue.

V. RESULTS

Table I provides the estimated parameters for our model over three time periods from the in-sample data set. We use these parameters to estimate the spot prices and the convenience yields at each trading date. Figures 9 and 10 illustrate the market and the term structures predicted by the model for 02/25/2009 and 02/03/2010. From the figures we see that the model accurately approximates both the long end and the short end of the term structure. In particular the spot prices are predicted accurately in accordance with the term structure, and the model also accommodates short-term effects like local steeping (increased curvature) of the term structure. Figure 11 plots the log of the futures prices of the first nearby contract (CL1 – 1 month expiry) with the log of the predicted value of the spot prices. From the figure, we see that estimates of spot price from our model closely follow the movement of the CL1 contract. Finally, Figure 12 plots the observed market volatility of the commodity prices with the model predicted volatility. The usual trend of the exponential decreasing volatilities is evident from the figure.

VI. CONCLUSIONS

The term structure of futures prices is an important indicator in the commodities futures market as it serves two pain purposes: a) to manage price risk, and b) to help in price discovery and market depth. In this paper, we propose a new three factor term structure model for one of the most developed commodity futures market - the crude oil market. Unlike existing three-factor models which have poor calibration performance arising from having to use both commodities and bond prices in the calibration process, our model relies only on commodities futures data for calibration. We develop an efficient point estimate procedure to calibrate our model and show that in spite of not being able to give us confidence intervals for estimates, the model performs on par with those based on Kalman filters. Since our model does not aggregate data and uses an iterative procedure to compute the estimates, it can be easily adapted
to obtain new parameter estimates in the presence of live data.

VII. FUTURE WORK

There are a few natural extensions to our work. An important observation is that there is some amount of redundancy in the parameters used in the model, as seen by the parameter values. We could incorporate market insight to decide which parameters to exclude from our model. Further reduction in complexity can also be achieved by building a parsimonious approximation to the convenience yield state variable. This model is quite tractable to be implemented on a commercial spreadsheet solver like Microsoft Excel, which allows seamless integration with other models and data used by the trading desks. An interesting exercise would be to extend our analysis to other important commodities markets like metals, gasoil, and naptha, to study the behavior of our model.

REFERENCES